Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. A. Behring

Exercise Sheet 8

Issue: 11.06. – Submission: 18.06. @ 10:00 Uhr – Discussion: 21./22.06.

Exercise 1: Scattering off a surface — 6 points

In the lectures we have calculated the cross section for scattering off a solid sphere. In this problem we consider elastic scattering of particles, coming from $z = -\infty$ with initial velocities parallel to the z-axis, off a solid object consisting of the points

$$\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; \sqrt{x^2 + y^2} \leq b \sin(\tfrac{z}{a}) \text{ and } 0 \leq z \leq \pi a\right\}. \tag{1.1}$$

In other words, the edge of the object is the surface of revolution around the z-axis generated by

$$\rho(z) = b \sin\left(\frac{z}{a}\right) \text{ für } 0 \leq z \leq \pi a\,. \tag{1.2}$$

The parameters a and b are positive.

(a) Make a sketch of this surface. Draw the path of a particle with impact parameter $0 < \rho < b$ and indicate the deflection angle θ. Demonstrate from the sketch that half of the deflection angle is related to the slope of the surface,

$$\tan\left(\frac{\theta}{2}\right) = \frac{\mathrm{d}\rho(z)}{\mathrm{d}z}\,. \tag{1.3}$$

(b) Use Eq. (1.3) to derive the relation between the impact parameter and the deflection angle,

$$\rho(\theta) = \sqrt{b^2 - a^2 \tan^2\left(\frac{\theta}{2}\right)}\,. \tag{1.4}$$

(c) Calculate the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = 2\pi\rho(\theta)\left|\frac{\mathrm{d}\rho(\theta)}{\mathrm{d}\theta}\right|\,. \tag{1.5}$$

(d) What are the minimal and maximal values of the scattering angle, θ_{min} and θ_{max}, corresponding to impact parameters $\rho \to b$ and $\rho \to 0$, respectively?

(e) Compute the total cross section

$$\sigma = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \mathrm{d}\theta\, \frac{\mathrm{d}\sigma}{\mathrm{d}\theta}\,. \tag{1.6}$$

Can you explain the simple result for σ from a geometrical point of view?

Exercise 2: Scattering in a $1/r^2$ potential $\boxed{\textbf{5 points}}$

In the lectures we have discussed scattering in a Coulomb potential $U(r) = \pm\alpha/r$ and derived the famous Rutherford formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \left(\frac{\alpha}{4E}\right)^2 \frac{2\pi}{\sin^4(\theta/2)}\,. \tag{2.1}$$

In this problem we will compute the cross section for scattering in a more strongly localised repulsive potential,

$$U(r) = \alpha/r^2 \qquad\qquad \text{with } \alpha > 0\,. \tag{2.2}$$

The scattering angle is given by

$$\theta = \pi - 2\rho \int_{r_{\min}}^{\infty} \frac{\mathrm{d}r}{r^2} \frac{1}{\sqrt{1 - \rho^2/r^2 - U(r)/E}} \tag{2.3}$$

The lower bound of integration $r_{\min}$ is equal to the distance between the origin and the turning point, see Fig. 1.

(a) Show that in this case $r_{\min} = \sqrt{\rho^2 + \alpha/E}$.

(b) Perform the integral in Eq. (2.3) and show that a particle with energy E and impact parameter ρ is deflected by an angle

$$\theta = \pi \left(1 - \frac{\rho}{\sqrt{\rho^2 + \alpha/E}}\right)\,. \tag{2.4}$$

(c) Invert Eq. (2.4) to obtain $\rho = \rho(\theta)$. Show that the differential scattering cross section, defined in Eq. (1.5) of the previous exercise, evaluates to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \frac{2\pi^3\alpha}{E} \frac{\pi - \theta}{\theta^2(2\pi - \theta)^2}\,. \tag{2.5}$$

(d) Sketch the differential cross section as a function of θ between $\theta = 0$ and $\theta = \pi$. How does the function behave as $\theta \to 0$? Is the total cross section finite, or divergent?

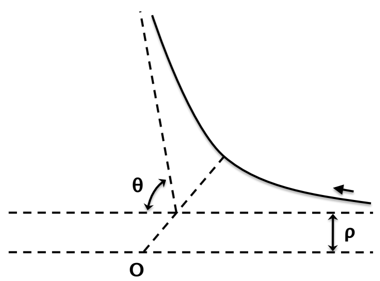

Figure 1: Definition of the impact parameter ρ, scattering angle θ.

Exercise 3: Particle capture

7 points

If this problem we are interested in the process where a particle comes from far away and flies towards the origin $r = 0$ as $t \to \infty$. In other words, the particle is captured by the central potential. We have already investigated on the previous exercise sheet under which conditions on the potential the particle can overcome the angular momentum barrier and reach the origin. Here we want to analyse this system again, now as a scattering problem.

(a) Assume that the particle flies along the z axis and express the angular momentum M in terms of the scattering parameter ρ.

Consider an attractive potential $U(r) = -\beta/r^2$ with $\beta > E\rho^2$.

(b) What are the minimal and maximal values of the impact parameter for which particle capture takes place? Calculate the total cross section for particle capture

$$\sigma = \int \mathrm{d}\sigma = \int_{\rho_{\min}}^{\rho_{\max}} \mathrm{d}\rho\, 2\pi\rho\,. \tag{3.1}$$

Hint: Remember the conditions that you derived in the last exercise on the previous sheet.

Consider now the potential $U(r) = \alpha/r - \beta/r^2$ with $\alpha > 0$ and $\beta > E\rho^2$.

(c) Sketch the effective potential $U_{\text{eff}}(r)$ in this case.

(d) Calculate the maximum value U_0 of the effective potential.

(e) For a particle to reach the origin, its energy must certainly exceed U_0. Show that this implies the following upper bound on the impact parameter:

$$\rho^2 < \frac{\beta}{E} - \frac{\alpha^2}{4E^2}\,. \tag{3.2}$$

(f) Using the fact that the impact parameter is positive, derive that

$$E > \frac{\alpha^2}{4\beta} \tag{3.3}$$

in order for a particle to reach the origin.

(g) Show that the total cross section for particle capture is given by

$$\sigma = \begin{cases} \pi\left(\frac{\beta}{E} - \frac{\alpha^2}{4E^2}\right) & \text{if } E > \frac{\alpha^2}{4\beta}\,, \\ 0 & \text{if } E < \frac{\alpha^2}{4\beta}\,. \end{cases} \tag{3.4}$$


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KEEP CALM
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IT'S NOT
ROCKET SCIENCE
WAS? Theoretische Teilchen-Physik - was ist das eigentlich? Ein Vortrag von Prof. Melnikov
WANN? Mittwoch, den 16.06.21 um 18:00 Uhr
WO? Zoom - Siehe dafür: https://fachschaft.physik.kit.edu
eine Veranstaltung des Mentorenprogramms