Karlsruhe Institute of Technology

SoSe 2021

Classical Theoretical Physics II

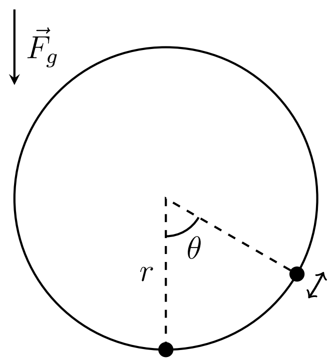
Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. A. Behring

Exercise Sheet 9

Issue: 18.06. – Submission: 25.06. @ 10:00 Uhr – Discussion: 28./29.06.

Exercise 1: Oscillations on a circle — 7 points

In this problem we will consider the setup depicted in sketch to the right. A particle with mass m and electrical charge q is free to move along the circle with radius r. It is affected by a gravitational force $\vec{F}_g = -mg\hat{e}_z$, and by a repulsive Coulomb force from the other particle (with charge q) that is fixed at the bottom of the circle. In order to make the formulae simpler, we use units such that the Coulomb potential between two particles of charge q a distance d apart, is q^2/d.



(a) Show that the Lagrangian for the particle may be given by

$$L = \frac{m}{2}r^2\dot{\theta}^2 + mgr\cos(\theta) - \frac{q^2}{2r\sin(\theta/2)}\,. \tag{1.1}$$

(b) Assume that $K = \frac{q^2}{8mgr^2} < 1$ such that the system has an equilibrium of the moving particle at an angle θ_0 between $\theta = 0$ and $\theta = \pi$. Find that equilibrium angle as a function of r, g, m and q.

(c) Choose a new coordinate $\xi = \theta - \theta_0$ and show that the close to the equilibrium angle θ_0 the Lagrangian can be approximated by

$$L = \frac{a}{2}\dot{\xi}^2 - \frac{b}{2}\xi^2\,. \tag{1.2}$$

Determine the parameters a and b.

(d) How can the new coordinate ξ be chosen differently such that the approximation of the Lagrangian becomes

$$L = \frac{m}{2}\dot{\xi}^2 - \frac{b'}{2}\xi^2\,? \tag{1.3}$$

(e) The movement may be described as small oscillations around the equilibrium position. Derive the equations of motion using the approximated Lagrangian in Eq. (1.2) or Eq. (1.3) and find the angular frequency ω of those oscillations.

Exercise 2: Driven oscillations

3 points

In this exercise we will consider the consequences of applying an external force to a harmonic oscillator. This is known as driven (or forced) oscillations.

(a) Forced oscillations are solutions to the equation

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \,. \tag{2.1}$$

As we have seen in the lectures, a particular solution is

$$x_p(t) = \int_{t_0}^{t} \frac{F(\tau)}{m\omega} \sin(\omega(t-\tau))\,\mathrm{d}\tau \,, \tag{2.2}$$

where t_0 is an arbitrary constant. This means that the general solution is given by

$$x(t) = A\sin(\omega t + \phi) + x_p(t) \,, \tag{2.3}$$

with A and ϕ being free parameters. Verify the validity of this solution, by direct insertion of Eq. (2.3) into Eq. (2.1).

(b) One commonly encountered case is a harmonic force $F(t) = F_0 \sin(\Omega t)$ with Ω in general being different from the eigenfreqency ω. Derive the general expression for $x(t)$ for this harmonic force. You may put $t_0 = 0$.

Exercise 3: Damped oscillations

4 points

In this exercise we will consider damped oscillations in one dimension. Damped oscillations are the result of movement in the force field

$$F = -kx - \mu\dot{x} \,. \tag{3.1}$$

The movement obeys the equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \,. \tag{3.2}$$

(a) For a particle with mass m, what are γ and ω_0 in terms of k, μ, and m?

(b) Whenever $\gamma < 2\omega_0$, the oscillations are called "weakly damped." Show that for that case, a solution to the equation of motion is

$$x = x_0 e^{-\gamma t/2} \cos(\omega_d t) \,, \tag{3.3}$$

with

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4} \,. \tag{3.4}$$

(c) What is the ratio between the amplitudes of successive swings of a weakly damped oscillator to the same side?

(d) When friction is present, the energy is not conserved. What is the energy of the weakly damped oscillator as a function of time? (Assume that $\gamma \ll \omega_0$)