Classical Theoretical Physics II

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Exercise Sheet 11

Issue: 02.07. – Submission: 09.07. @ 10:00 Uhr – Discussion: 12./13.07.

Exercise 1: Hamiltonians of various systems — 5 points

Find the canonical momenta, the Hamiltonian and Hamilton's equations of motion for each of the following cases.

(a) A free particle with mass m in one dimension.

(b) A harmonic oscillator with mass m and angular frequency ω in one dimension.

(c) A particle with mass m in the one-dimensional potential $U(x) = \alpha x^n$.

(d) A particle with mass m in a three-dimensional potential $U(r) = -k/r$. Use polar coordinates r, θ, ϕ.

(e) Two particles with masses m and M moving in a two-dimensional plane and interacting gravitationally. Use Cartesian coordinates x_i, y_i for each particle. (What could have been a better set of coordinates for this problem?)

Exercise 2: Runge-Lenz vector and Poisson brackets — 5 points

Consider a particle with mass m, position $\vec{r}$ and momentum $\vec{p}$ in three-dimensional space. The angular momentum of the particle is given by $\vec{M} = \vec{r} \times \vec{p}$.

(a) Calculate the Poisson brackets

$$\{M_i, r_j\}\,, \qquad \{M_i, p_j\}\,, \qquad \{M_i, M_j\}\,, \qquad \{M_i, M^2\}\,. \tag{2.1}$$

Make use of the properties of the Poisson brackets that were introduced in the lectures in Chapter 9 in Eqs. (20) to (22).

(b) The Runge-Lenz vector is given by $\vec{A} = \vec{p} \times \vec{M} - mk\hat{r}$, where $\hat{r} = \vec{r}/r$ is the unit vector in radial direction. Prove using Poisson brackets that the Runge-Lenz vector is conserved in the Kepler problem. That is, show that $\{H, A_i\} = 0$ holds for the Hamiltonian $H = \frac{p^2}{2m} - \frac{k}{r}$.

(c) We have already shown in a previous problem that the Runge-Lenz vector is no longer conserved if we add a r^{-2} in addition to the r^{-1} term in the potential. Consider the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{k}{r} + \frac{C}{r^2} \tag{2.2}$$

and calculate the time derivative of the Runge-Lenz vector using Poisson brackets and compare your result to the result of problem 8.2 (d).

Exercise 3: Particle in magnetic field **9 points**

The motion of a particle with mass m and charge e is described by the Lagrangian

$$L = \frac{m}{2}\dot{\vec{r}}^2 - \left(e\phi(\vec{r},t) - \frac{e}{c}\vec{A}(\vec{r},t)\cdot\dot{\vec{r}}\right) , \tag{3.1}$$

where ϕ and $\vec{A}$ are the electromagnetic potentials.

(a) Determine the canonical momentum $\vec{p}$, construct the Hamiltonian H and show that Hamilton's equations are given by

$$\dot{r}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right) , \qquad \dot{p}_i = \frac{e}{mc}\left(p_j - \frac{e}{c}A_j\right)\frac{\partial A_j}{\partial r_i} - e\frac{\partial \phi}{\partial r_i} , . \tag{3.2}$$

(b) Derive the Lorentz force law from Hamilton's equations. *Hint:* use index notation and the following expressions for the electric and magnetic fields in terms of the electromagnetic potentials

$$E_i = -\frac{\partial \phi}{\partial r_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} , \qquad B_i = \epsilon_{ijk}\frac{\partial A_k}{\partial r_j} . \tag{3.3}$$

(c) In the presence of a magnetic field we have that $\vec{p} \neq m\vec{v}$, where $\vec{v} = \dot{\vec{x}}$. This is evident from the first of Hamilton's equations in Eq. (3.2). Their Poisson brackets also reflect this fact: while $\{p_i, p_j\} = 0$, show that

$$\{v_i, v_j\} = -\frac{e}{m^2 c}\epsilon_{ijk}B_k . \tag{3.4}$$

(d) Consider the specific case of a particle restricted to the x, y-plane, with potentials $\phi(\vec{r},t) = 0$ and $\vec{A}(\vec{r},t) = (-By, 0, 0)^T$. What are the associated $\vec{E}$ and $\vec{B}$ fields? Give Hamilton's equations for this case. Show that they imply

$$m\dot{y} + \frac{eB}{c}x = K_1 , \qquad m\dot{x} - \frac{eB}{c}y = K_2 , \tag{3.5}$$

with unknown constants $K_{1,2}$.

(e) Show that the general solution to Eq. (3.5) is given by

$$x(t) = \frac{K_1}{m\omega} + R\sin(\omega t + \phi) , \tag{3.6}$$

$$y(t) = -\frac{K_2}{m\omega} + R\cos(\omega t + \phi) , \tag{3.7}$$

where $\omega = \frac{eB}{mc}$ and R and ϕ are (for now) unknown integration constants.

(f) Impose the boundary conditions $x(0) = x_0$, $y(0) = y_0$, $\dot{x}(0) = v_0$ and $\dot{y}(0) = 0$, and sketch the motion of the particle.