Classical Theoretical Physics II

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Exercise Sheet 12

Issue: 09.07. – Submission: 16.07. @ 10:00 Uhr – Discussion: 19./20.07.

Exercise 1: Anharmonic oscillator (5 points)

It was shown in the lectures that the relation between the old and new coordinates and momenta under a canonical transformation $\{p, q\} \to \{P, Q\}$ can be compactly described by the generating function $F(q, Q, t)$, where

$$\frac{\partial F}{\partial q} = p\,, \qquad \frac{\partial F}{\partial Q} = -P\,, \qquad \frac{\partial F}{\partial t} = K - H\,. \tag{1.1}$$

The dependence of the generating function on q and Q as the independent variables is not the only possible choice. It is also possible to use generating functions $F_2(q, P, t)$, $F_3(p, Q, t)$ and $F_4(p, P, t)$. In this exercise we will consider an anharmonic oscillator, which is described by the Hamiltonian

$$H(p, q) = \frac{p^2}{2m}(1 + \epsilon\beta q) + \frac{1}{2}m\omega^2 q^2(1 + \epsilon\alpha q)\,, \tag{1.2}$$

where ϵ is a small dimensionless constant. We will now study a canonical transformation that is described by the generating function $F_2(q, P, t)$.

(a) For the generating function $F(q, Q, t)$ the vanishing of the variation of the action implies the following condition:

$$\mathrm{d}F(q, Q, t) = p\,\mathrm{d}q - P\,\mathrm{d}Q + (K - H)\,\mathrm{d}t\,, \tag{1.3}$$

where $K(P, Q)$ is the Hamiltonian for the new coordinates. Add the differential $\mathrm{d}(PQ)$ to both sides of the above equation and rewrite the right-hand side as a differential of a function depending on the variables q, P, and t. Identify the left-hand side as $\mathrm{d}F_2(q, P, t)$ and derive the relations

$$\frac{\partial F_2}{\partial q} = p\,, \qquad \frac{\partial F_2}{\partial P} = Q\,, \qquad \frac{\partial F_2}{\partial t} = K - H \tag{1.4}$$

from it.

(b) Which transformation is generated by $F_2(q, P, t) = qP$?

(c) Let us now consider the generating function

$$F_2(q, P, t) = qP + \epsilon a q^2 P + \epsilon b P^3\,. \tag{1.5}$$

Determine values for a and b such that $K(Q, P)$ agrees with the Hamiltonian for a *harmonic* oscillator up to and including terms of order ϵ.

(d) Express $q(t)$ in terms of the known sinusoidal solutions of the harmonic oscillator.

Exercise 2: Areas in phase space

6 points

Let us consider, again, the harmonic oscillator given by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m q^2 \omega^2 \,. \tag{2.1}$$

(a) Verify that

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \theta_0) \,, \qquad p = \sqrt{2Em} \cos(\omega t + \theta_0) \,, \tag{2.2}$$

are solutions to Hamilton's equations.

(b) Draw the path of an oscillation in (q, p) phase space, and indicate on the drawing the dependence on E and t.

(c) Let m and ω be known exactly. The energy is only known to lie in the inteval between E_0 and $E_0 + \mathrm{d}E$ and the time to be betweeen t_0 and $t_0 + \mathrm{d}t$. Consider the mapping $\{p, q\} \to \{E, t\}$ and find the area of (q, p) phase space in which the particle may be found as a function of $\mathrm{d}E$ and $\mathrm{d}t$.
Hint: Consider how infinitesimal area elements transform under variable changes.

(d) We will now do a canonical transformation to new variables Q and P, defined by the generating function

$$F = \frac{m\omega}{2} q^2 \cot(Q) \,, \tag{2.3}$$

(Here, cot is the co-tangent $\cot(z) = \cos(z)/\sin(z)$.)
Find expressions for q and p in terms of Q and P.

(e) Find expressions for Q and P.

(f) Draw the path of an oscillation in (Q, P) phase space, and determine the area in that phase space in which the particle may be found if it has the $\mathrm{d}E$ and $\mathrm{d}t$ uncertainties discussed above.

Exercise 3: Action as a generating function

6 points

The motion of a particle in a homogeneous gravitational field is subject to the Hamiltonian

$$H = \frac{p^2}{2m} + mgq \tag{3.1}$$

and the solutions to Hamilton's equations are given by

$$q(t) = -\frac{1}{2}gt^2 + \frac{P}{m}t + Q \qquad \text{und} \qquad p(t) = P - mgt\,, \tag{3.2}$$

where (Q, P) parametrises the initial conditions at time $t = 0$.

(a) For a fixed point in time t we can interpret the solutions of Hamilton's equations as a transformation which associates to the initial conditions (Q, P) the coordinate and momentum at time t, i.e. $(q(t), p(t))$. Is this transformation canonical?

Next, we want to consider the opposite problem, that is, given the solution (q, p), can we directly find a generating function $S(q, P, t)$ of a canonical transformation that transforms those solutions into the initial conditions?

(b) Starting from the Hamiltonian $H(q, p)$, find the velocity $\dot{q}(q, p)$ and the Lagrangian $L(q, \dot{q})$.

(c) Take the solutions of the equations of motion and calculate the action

$$S = QP + \int_0^t \mathrm{d}t'\, L(q(t'), \dot{q}(t'))\,. \tag{3.3}$$

Express the result as a function of $q = q(t)$, P and t.

(d) Use $S(q, P, t)$ as a generating function of a canonical transformation from (q, p) to (Q, P). What is the new Hamiltonian $K(Q, P)$? Interpret the result.