Vektoroperatoren in zylindrischen Koordinaten

Für $f = f(\rho, \varphi, z)$ und $\vec{A} = A_\rho(\rho, \varphi, z)\,\hat{\mathrm{e}}_\rho + A_\varphi(\rho, \varphi, z)\,\hat{\mathrm{e}}_\varphi + A_z(\rho, \varphi, z)\,\hat{\mathrm{e}}_\mathrm{z}$:

$$\begin{aligned}
\vec{\nabla} f &= \frac{\partial f}{\partial \rho}\,\hat{\mathrm{e}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\,\hat{\mathrm{e}}_\varphi + \frac{\partial f}{\partial z}\,\hat{\mathrm{e}}_\mathrm{z} \\
\Delta f &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \\
\vec{\nabla} \cdot \vec{A} &= \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \vec{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\mathrm{e}}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\mathrm{e}}_\varphi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathrm{e}}_\mathrm{z}
\end{aligned}$$

Differentialoperatoren in Kugelkoordinaten

Für $f = f(r, \theta, \varphi)$ und $\vec{A} = A_r(r, \theta, \varphi)\,\hat{\mathrm{e}}_\mathrm{r} + A_\theta(r, \theta, \varphi)\,\hat{\mathrm{e}}_\theta + A_\varphi(r, \theta, \varphi)\,\hat{\mathrm{e}}_\varphi$:

$$\begin{aligned}
\vec{\nabla} f &= \frac{\partial f}{\partial r}\,\hat{\mathrm{e}}_\mathrm{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\mathrm{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\,\hat{\mathrm{e}}_\varphi \\
\Delta f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \\
\vec{\nabla} \cdot \vec{A} &= \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi} \\
\nabla \times \vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\varphi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right)\hat{\mathrm{e}}_\mathrm{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial(r A_\varphi)}{\partial r}\right)\hat{\mathrm{e}}_\theta + \frac{1}{r}\left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\mathrm{e}}_\varphi
\end{aligned}$$

Kugelflächenfunktionen

Die Kugelflächenfunktionen sind gegeben durch

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell + 1}{4\pi}\frac{(\ell - m)!}{(\ell + m)!}}\, P_\ell^{(m)}(\cos\theta) e^{im\varphi}$$

mit den assoziierten Legendre-Polynomen

$$P_\ell^{(m)} = (-1)^m (1 - x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_\ell(x)\ .$$

Es gilt

$$\begin{aligned}
\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y_{\ell m}}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\ell m}}{\partial \varphi^2} + \ell(\ell + 1) Y_{\ell m} &= 0\,, \\
\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, Y^*_{\ell' m'}(\theta, \varphi) Y_{\ell m}(\theta, \varphi) &= \delta_{\ell\ell'}\delta_{mm'}\ , \\
\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi) &= \delta(\cos\theta' - \cos\theta)\delta(\varphi' - \varphi)\ .
\end{aligned}$$