Klinkhamer / Rupp
WS 2004/2005

1) Coulomb-Feld: $\vec{E}(\vec{x}) = q \cdot \frac{\vec{x}}{|\vec{x}|^3}$ $\quad \vec{B} = B \cdot \vec{e}_z$

$$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) = \frac{q^2}{8\pi |\vec{x}|^2} + \frac{B^2}{8\pi}$$

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) = \frac{c \cdot B \cdot q}{4\pi |\vec{x}|^2} \underbrace{\vec{e}_r \times \vec{e}_z}_{-\sin\vartheta \cdot \vec{e}_\varphi} = -\frac{c \cdot B \cdot q}{4\pi r^2} \cdot \sin\vartheta \, \vec{e}_\varphi$$

2) a) $\rho(\vec{x}) = \delta(z) \cdot \delta(\sqrt{x^2+y^2} - a) \cdot \lambda_0 \cdot \cos\varphi$

$$= \delta(r \cdot \cos\vartheta)\, \delta(r \cdot \sin\vartheta - a) \cdot \lambda_0 \cdot \cos\varphi = \frac{1}{r}\delta(\cos\vartheta)\,\delta(r-a) \cdot \lambda_0 \cdot \cos\varphi$$

$$q_{lm} = \int dr \, d\cos\vartheta \, d\varphi \, r^2 \rho(r,\vartheta,\varphi)\, Y^*_{lm}(\vartheta,\varphi)\, r^l$$

$$= \int d\varphi \, a^2 \cdot \frac{1}{a} \lambda_0 \cos\varphi \cdot a^l \, Y^*_{lm}\left(\frac{\pi}{2}, \varphi\right)$$

$$Y^*_{lm} = \sqrt{\frac{(2l+1)\cdot(l-m)!}{4\pi \, (l+m)!}}\, P_l^m(\cos\vartheta) \cdot e^{-im\varphi}$$

$$\int d\varphi \cos\varphi \cdot e^{-im\varphi} = \int d\varphi \, \frac{1}{2}(e^{i\varphi} + e^{-i\varphi}) \cdot e^{-im\varphi}$$

$$= \frac{1}{2}\left[\frac{1}{-i(m-1)} e^{-i(m-1)\varphi} + \frac{1}{-i(m+1)} e^{-i(m+1)\varphi}\right]$$

$$= \frac{1}{2}(\delta_{m1} + \delta_{m-1}) \cdot 2\pi$$

$$q_{lm} = \pi(\delta_{m1} + \delta_{m-1}) \cdot \sqrt{\frac{(2l+1)(l-m)!}{4\pi \, (l+m)!}} \cdot a^{l+1} \cdot \lambda_0 \cdot P_l^m(0)$$

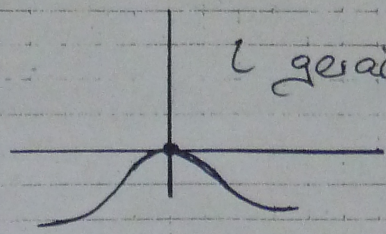
$$m = +1: \quad P_l^1(0) = -\sqrt{1-0}\, \frac{d}{dx}P_l(x)\Big|_{x=0} = -P_l'(0)$$

$$q_{l1} = -\pi \cdot \sqrt{\frac{(2l+1)\,(l-1)!}{4\pi \, (l+1)!}} \cdot a^{l+1} \cdot \lambda_0 \cdot P_l'(0)$$

$$m = -1: \quad P_l^{-1}(0) = -\frac{(l-1)!}{(l+1)!} P_l^1(0) = +\frac{(l-1)!}{(l+1)!} P_l'(0)$$

$$q_{l-1} = \pi \sqrt{\frac{(2l+1)(l+1)!}{4\pi \, (l-1)!}}\, a^{l+1} \cdot \lambda_0 \, \frac{(l-1)!}{(l+1)!} P_l'(0) = -q_{l1}$$

b) $P_l(-x) = (-1)^l \cdot P_l(x)$



l gerade $\Rightarrow P_l'(0) = 0$

(5) Lorentz-Eichung: $\partial_\mu A^\mu = 0$

a) $\vec{A}(\vec{x},t) = \vec{A}_0 \cdot e^{-i\vec{k}\vec{x} - i\omega t}$ $\qquad \omega = c \cdot |\vec{k}|$

$(A^\mu) = (\phi, \vec{A})$

$\frac{1}{c}\partial_t \phi + \nabla \vec{A} = 0 \Rightarrow \partial_t \phi = -c \cdot \nabla \vec{A} = -ci\vec{k}\cdot\vec{A}_0 \cdot e^{-i\vec{k}\vec{x} - i\omega t}$

$\Rightarrow \phi(t) = \frac{c \cdot \vec{k}\cdot\vec{A}_0}{\omega} \cdot e^{-i\vec{k}\vec{x} - i\omega t}$

$A^\mu = a^\mu \cdot e^{-k_\nu x^\nu} = a^\mu \exp(-ik^0 c t + i\vec{k}\vec{x})$

$k^\nu = (k^0, \vec{k}) = (\frac{\omega}{c}, \vec{k}) \qquad a^\mu = (\frac{c\vec{k}\cdot\vec{A}_0}{\omega}, \vec{A}_0)$

b) $k_\mu A^\mu = i\partial_\mu A^\mu = 0$

$A_\mu A^\mu = (\frac{c^2}{\omega^2}(\vec{k}\cdot\vec{A}_0)^2 - \vec{A}_0^2) \cdot e^{-2ik_\nu x^\nu}$

$k_\mu k^\mu = \frac{\omega^2}{c^2} - \vec{k}^2 = 0$

c) $T^{\mu\nu} = \frac{1}{4\pi}(F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma})$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -\underline{ik^\mu} A^\nu + ik^\nu A^\mu$

$F^\mu{}_\lambda F^{\lambda\nu} = (-ik^\mu A_\lambda + ik_\lambda A^\mu)(-ik^\lambda A^\nu + ik^\nu A^\lambda)$

$= k^\mu k^\nu A^\lambda A_\lambda$.

Seite

$$F_{\nu\lambda}F^{\nu\lambda} = \eta_{\mu\nu} F^{\mu}{}_{\lambda} F^{\lambda\nu} = k_\mu k^\mu A_\lambda A^\lambda = 0$$

$$\Rightarrow T^{\mu\nu} = \frac{1}{4\pi} k^\mu k^\nu a_\lambda a^\lambda \cdot e^{-2ik_\rho x^\rho}$$

d) $A_\mu \Rightarrow A'_\mu = A_\mu + \partial_\mu \chi$

$$\partial^\mu A'_\mu = 0 \Rightarrow \Box \chi = 0$$

$$\chi = \chi_0 \cdot e^{-ik_\nu x^\nu} \qquad \partial_\mu \chi = -ik_\mu \chi_0 \cdot e^{-ik_\nu x^\nu}$$

$$\Rightarrow \vec{A}' = (\vec{A}_0 + i\vec{k}\chi_0)\cdot e^{-i\vec{k}\vec{x}}$$

$$\chi_0 = i \cdot \frac{\vec{A}_0\cdot\vec{k}}{\vec{k}^2} \Rightarrow \vec{A}' = \left(\vec{A}_0 - \frac{\vec{A}_0\cdot\vec{k}}{\vec{k}^2}\vec{k}\right)\cdot e^{-i\vec{k}\vec{x}}$$

$$\vec{k}\cdot\vec{A}' = \vec{k}\cdot\vec{A}_0 - \frac{\vec{A}_0\cdot\vec{k}}{\vec{k}^2}\vec{k}\cdot\vec{k} = 0$$

3) a) $\vec{r}(t) = (0, 0, -\frac{1}{2}gt^2)$

$$\rho(\vec{x},t) = q\cdot\delta(\vec{x}-\vec{r}(t)) = q\cdot\delta(x)\cdot\delta(y)\,\delta(z+\tfrac{1}{2}gt^2)$$

$$\vec{j}(\vec{x},t) = \rho\cdot\dot{\vec{r}}(t) = -q\cdot g t\,\delta(x)\cdot\delta(y)\cdot\delta(z+\tfrac{1}{2}gt^2)$$

b) $\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} = -\vec{e}_z \frac{g\cdot t\cdot q}{c\cdot\sqrt{x^2+y^2+(z+\frac{1}{2}gt^2)^2}}$

c) $\phi_m = \int \vec{B}\cdot d\vec{S} = \int (\nabla\times\vec{A})\,d\vec{S} = \int \vec{A}\,d\vec{l} = 0$

da $\vec{A} \perp d\vec{l}$

4) $\phi = q \cdot \frac{e^{-r/\lambda}}{r}$

a) $\Delta\phi = -4\pi\rho$

$$\Delta\phi = \frac{1}{r^2}\partial_r(r^2\partial_r\phi) \Rightarrow \rho = \frac{q \cdot e^{-r/\lambda}}{4\pi\lambda^2 \cdot r} \qquad r \neq 0$$

$$\vec{E} = -\nabla\phi = \left(\frac{1}{\lambda} + \frac{1}{r}\right) \cdot \phi \cdot \vec{e}_r$$

$$\int \vec{E}\, d\vec{S} = 4\pi R^2 \left(\frac{1}{\lambda} + \frac{1}{R}\right) \cdot \phi(R) \xrightarrow{R \to 0} 4\pi q$$

$$\Rightarrow \rho = \frac{-q \cdot e^{-r/\lambda}}{4\pi\lambda^2 \cdot r} + q \cdot \delta(\vec{x})$$

b) $Q(R) = q \cdot e^{-R/\lambda}\left(1 + \frac{R}{\lambda}\right)$

c) $\nabla \times \vec{E} = -\nabla \times (\nabla\phi) = 0$