A1

a) $$\partial_\alpha \big(\overset{(+*)}{\partial_\beta A_\gamma} - \overset{(-\#)}{\partial_\gamma A_\beta}\big) + \partial_\gamma \big(\overset{(+\#)}{\partial_\alpha A_\beta} - \overset{(-\square)}{\partial_\beta A_\alpha}\big) + \partial_\beta \big(\overset{(+\square)}{\partial_\gamma A_\alpha} - \overset{(-*)}{\partial_\alpha A_\gamma}\big) = 0$$

(vert. d. part. Ableitungen)

b) $$E_1 = E_2 = \frac{Mc^2}{2} = \sqrt{m^2c^4 + |\vec{p}|^2c^2} \Rightarrow |\vec{p}|^2 = \frac{M^2c^2}{4} - m^2c^2$$

$$\Leftrightarrow \quad |\vec{p}| = c\sqrt{\frac{M^2}{4} - m^2} \quad \left(m < \frac{M}{2} \right) \quad \Bigg| \quad M = 2\sqrt{m^2 + |\vec{p}|^2/c^2}$$

c) $$\vec{B} = \vec{\nabla} \times \vec{A} = \left(\partial_x A_y - \partial_y A_x \right) \vec{e}_z \, , \quad A_z = f(z)$$

$$\Rightarrow \quad A_x = -\frac{1}{2} y^2 b_0 \, , \quad A_y = \frac{1}{2} x^2 b_0 \, , \quad \vec{\nabla} \cdot \vec{A} = \partial_z A_z = f'(z) \neq 0$$

d) $$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \, , \quad \epsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\rho\sigma} = 0$$

oder: $$\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu} = 0$$

e) $$\Delta \varphi(\vec{r}) = \int d^3r' \, \Delta_{\vec{r}} \, G(\vec{r} - \vec{r}') \, \rho(\vec{r}') = \int d^3r' \left(-\frac{1}{\epsilon_0} \delta(\vec{r} - \vec{r}') \right) \rho(\vec{r}')$$

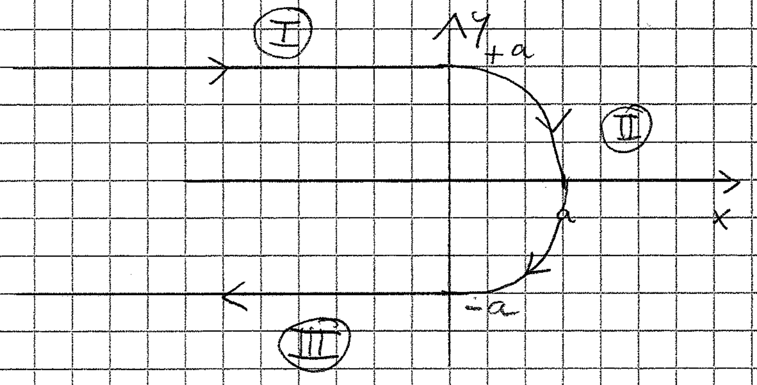
$$= -\frac{\rho(\vec{r})}{\epsilon_0} = \Delta \varphi(\vec{r}) \quad \checkmark$$

f) $$a' \cdot b' = \underbrace{\Lambda^\rho{}_\mu a^\mu}_{a'^\rho} \underbrace{\Lambda^\sigma{}_\nu b^\nu}_{b'^\sigma} g_{\rho\sigma} = a^\mu \underbrace{\Lambda^\rho{}_\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu}_{(\Lambda^T g \Lambda)_{\mu\nu} = g_{\mu\nu}} b^\nu$$

$$= a^\mu b^\nu g_{\mu\nu} = a \cdot b \quad \checkmark$$

A2

a)



, $z = 0$

b) $\vec{B}(\vec{r}_0) = \vec{B}_I + \vec{B}_{II} + \vec{B}_{III}$ (Lineare Superposition),

Berechnung mit Biot-Savart

I : $y' = a$, $z' = 0$, $\vec{r}\,' = x'\vec{e}_x + a\vec{e}_y$, $\vec{r}_0 = (0,0,0)^T$

$$\vec{j}(\vec{r}\,') \times (\vec{r}_0 - \vec{r}\,') = I\,\delta(z')\,\delta(y'-a)\,\vec{e}_x \times (-a\vec{e}_y)$$

$$= -I\,\delta(z')\,\delta(y'-a)\,a\,\vec{e}_z$$

$$\Rightarrow \vec{B}_I(\vec{r}_0) = -\frac{\mu_0 I}{4\pi}\int_{-\infty}^{0} dx'\,\frac{a\vec{e}_z}{\sqrt{x'^2 + a^2}^{\,3}} = -\frac{\mu_0 a I}{4\pi a^2}\,\vec{e}_z$$

$$= -\frac{\mu_0 I}{4\pi a}\,\vec{e}_z$$

$\vec{e}_{\varphi'} \times \vec{e}_{\rho'} = -\vec{e}_z$

II : $\vec{r}\,' = a\vec{e}_{\rho'}$, $\vec{j}(\vec{r}\,') \times (\vec{r}_0 - \vec{r}\,') = -I\,\vec{e}_{\varphi'} \times (-a\vec{e}_{\rho'})\,\delta(z')\,\delta(\rho'-a)$

$$= -I a\,\vec{e}_z\,\delta(z')\,\delta(\rho'-a)$$

$$\Rightarrow \vec{B}_{II}(\vec{r}_0) = -\frac{\mu_0 I}{4\pi}\int_{-\pi/2}^{\pi/2} d\varphi'\,\frac{a^2\vec{e}_z}{a^3} = -\frac{\mu_0 I}{4\pi a}\,\pi\,\vec{e}_z$$

III : $\vec{r}\,' = x'\vec{e}_x - a\vec{e}_y$, $\vec{j}(\vec{r}\,') \times (\vec{r}_0 - \vec{r}\,') = -I a\,\vec{e}_z\,\delta(z')\,\delta(y'+a)$

$\Rightarrow \vec{B}_{III} = \vec{B}_I$, (wg. $|\vec{r}_0 - \vec{r}\,'| = \sqrt{a^2 + x'^2}$)

Also ist $\vec{B}(\vec{r}_0) = -\dfrac{\mu_0 I}{4\pi a}(2 + \pi)\,\vec{e}_z$

A31 $\vec{p} = \vec{p}(\tau),\ \tau = t - r/c;\ \frac{\partial \vec{p}}{\partial t} = \underbrace{\frac{\partial \vec{p}}{\partial \tau}}_{\dot{\vec{p}}} \overbrace{\frac{\partial \tau}{\partial t}}^{1} = \dot{\vec{p}},\ \frac{\partial \vec{p}}{\partial x} = \dot{\vec{p}}\, \frac{\partial \tau}{\partial x}\ \left(\dot{\vec{p}} = \frac{\partial \vec{p}}{\partial \tau}\right)$

a) $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{e}_x\, \partial_y A_z - \vec{e}_y\, \partial_x A_z \quad ; \quad \ddot{p}_z \equiv (\ddot{\vec{p}})_z = -\omega^2 p_0 \cos(\,)$

$$= \vec{e}_x \frac{\mu_0}{4\pi r} \ddot{p}_z \left(-\frac{y}{rc}\right) + \vec{e}_y \frac{\mu_0}{4\pi r} \ddot{p}_z \left(\frac{x}{rc}\right) + O(1/r^2)$$

$$= \frac{\mu_0}{4\pi r^2 c} \underbrace{\ddot{p}_z \left(x \vec{e}_y - y \vec{e}_x\right)}_{-\vec{r} \times \ddot{\vec{p}}} \left(= -\frac{\mu_0}{4\pi c} \frac{\vec{r} \times \ddot{\vec{p}}}{r^2} \right) + O(1/r^2)$$

Hierbei verwendet: $\partial_x (1/r) = -\frac{x}{r^3} = O(1/r^2) \to 0$

$$\partial_x \partial_t \vec{p}(t - r/c) = \underbrace{\left(\partial^2 t\, \vec{p}(t - r/c)\right)}_{\equiv \ddot{\vec{p}}} \cdot \left(-\frac{x}{rc}\right), \quad \dot{p}_z = O(1/r^2)$$

b) $\vec{E} = -\vec{\nabla}\varphi - \dot{\vec{A}}, \quad -\dot{\vec{A}} = -\frac{\mu_0}{4\pi r} \ddot{\vec{p}}(t - r/c),$

$\varphi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{cr^2}\, z\, \dot{p}_z(t - r/c)$; In $O(1/r)$ gilt:

$$(-\vec{\nabla}\varphi)_x = \frac{1}{4\pi\epsilon_0} \frac{1}{cr^2}\, z\, \ddot{p}_z \left(\frac{x}{rc}\right); \quad (-\vec{\nabla}\varphi)_y = \frac{1}{4\pi\epsilon_0} \frac{z}{cr^2} \left(\frac{y}{rc}\right) \ddot{p}_z$$

$$(-\vec{\nabla}\varphi)_z = \frac{1}{4\pi\epsilon_0} \frac{1}{cr^2} \frac{z^2}{rc} \ddot{p}_z$$

$$\Rightarrow -\vec{\nabla}\varphi = \frac{1}{4\pi\epsilon_0} \frac{1}{c^2 r^3} \begin{pmatrix} +zx \\ +zy \\ +z^2 \end{pmatrix} \ddot{p}_z + O(1/r^2)$$

Also ist $\vec{E} = \frac{\ddot{p}_z}{4\pi\epsilon_0 c^2 r^3} \begin{pmatrix} zx \\ zy \\ z^2 - r^2 \end{pmatrix} + O(1/r^2) \quad \left(\text{mit } \mu_0 \epsilon_0 = \frac{1}{c^2}\right)$

$$\left(\begin{aligned} &= \frac{1}{4\pi\epsilon_0 c^2} \left[\vec{r}\,(\ddot{\vec{p}} \cdot \vec{r}) - \ddot{\vec{p}}\,(\vec{r} \cdot \vec{r}) \right] \frac{1}{r^3} \\ &= \frac{1}{4\pi\epsilon_0} \left[-\frac{(\vec{r} \times \ddot{\vec{p}}) \times \vec{r}}{r^3 c^2} \right] + O(1/r^2) \end{aligned} \right)$$

A.4 a) Aus Maxwell-Gl:

$$\vec\nabla\cdot\vec E=0 \qquad \vec\nabla\cdot\vec B=0$$
$$\vec\nabla\times\vec E=-\dot{\vec B} \qquad \vec\nabla\times\vec B=\frac{1}{c^2}\dot{\vec E}$$

$$\Rightarrow \vec\nabla\times(\vec\nabla\times\vec E)=\vec\nabla(\underbrace{\vec\nabla\cdot\vec E}_{=0})-\Delta\vec E=-\vec\nabla\times\dot{\vec B}$$

$$\Rightarrow -\Delta\vec E=-\frac{1}{c^2}\ddot{\vec E} \Rightarrow \Box\vec E=0.$$

Analog $\Box\vec B=0$

Aus $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec E=0 \Rightarrow$

$$\left[\frac{(-i\omega)^2}{c^2}-\left(-\left(\frac{\pi}{a}\right)^2+(ik)^2\right)\right]\vec E=0 \Rightarrow \boxed{\frac{\omega^2}{c^2}=\frac{\pi^2}{a^2}+k^2} \qquad (4p)$$

$$-\omega^2+\frac{\pi^2}{a^2}+k^2=0$$

b) $\mathrm{Re}\,\vec E=E_0\sin\left(\frac{\pi x}{a}\right)\cos(kz-\omega t)\,\vec e_y=E_0 S_x C_z\vec e_y$, $\begin{cases}S_x=\sin\left(\frac{\pi x}{a}\right)\\ C_z=\cos(kz-\omega t)\end{cases}$

$\mathrm{Re}\,\vec B=\frac{E_0}{\omega}\left[-k S_x C_z\vec e_x+\frac{\pi}{a}C_x S_z\vec e_z\right]$ $\begin{cases}C_x=\cos\left(\frac{\pi x}{a}\right)\\ S_z=\sin(kz-\omega t)\end{cases}$

$$\vec S=\frac{1}{\mu_0}(\mathrm{Re}\vec E)\times(\mathrm{Re}\vec B)=$$
$$=\frac{1}{\mu_0}\frac{E_0^2}{\omega}\left[k S_x^2C_z^2\vec e_z+\frac{\pi}{a}S_xC_xS_zC_z\vec e_x\right] \qquad (2p)$$

$$w=\frac{1}{2\mu_0}\left[|\mathrm{Re}\vec B|^2+\frac{1}{c^2}|\mathrm{Re}\vec E|^2\right]=$$
$$=\frac{1}{2\mu_0}\frac{E_0^2}{\omega^2}\left[k^2S_x^2C_z^2+\left(\frac{\pi}{a}\right)^2C_x^2S_z^2+\frac{\omega^2}{c^2}S_x^2C_z^2\right] \qquad (2p)$$

Bem: $\vec\nabla\cdot\vec S+\frac{\partial w}{\partial t}=0$ (Energieerhaltung)

$$\vec\nabla\cdot\vec S=\frac{E_0^2}{\mu_0}\frac{1}{\omega}\left[k^2S_x^2\,2C_z(-S_z)+\left(\frac{\pi}{a}\right)^2(C_x^2-S_x^2)S_zC_z\right]\overset{(a)}{=}$$
$$=\frac{E_0^2}{\mu_0}\frac{1}{\omega}\left[-2k^2S_x^2+\left(\frac{\omega^2}{c^2}-k^2\right)(C_x^2-S_x^2)\right]S_zC_z$$
$$=\frac{E_0^2}{\mu_0\omega}\left[\frac{\omega^2}{c^2}(C_x^2-S_x^2)-k^2\right] \qquad (*)$$

$$\frac{\partial w}{\partial t}=\frac{1}{2\mu_0}\frac{E_0^2}{\omega^2}\cdot 2\omega\left[k^2S_x^2C_z(-S_z)(-1)+\frac{\pi^2}{a^2}C_x^2S_zC_z(-1)+\frac{\omega^2}{c^2}S_x^2C_z(-S_z)(-1)\right]$$
$$\overset{(a)}{=}\frac{E_0^2}{\mu_0\omega}\left[k^2S_x^2-\left(\frac{\omega^2}{c^2}-k^2\right)C_x^2+\frac{\omega^2}{c^2}S_x^2\right]C_zS_z$$
$$=\frac{E_0^2}{\mu_0\omega}\left[k^2-\frac{\omega^2}{c^2}(C_x^2-S_x^2)\right] \qquad (**)$$

Aus (*) + (**) $\Rightarrow$ $\vec\nabla\cdot\vec S+\frac{\partial w}{\partial t}=0.$