1) $\vec{\nabla}\cdot(\vec{V}\times\vec{W}) =$

a) $\partial_i \epsilon_{ijk} V_j W_k = \epsilon_{ijk}\left[(\partial_i V_j) W_k + V_j(\partial_i W_k)\right]$ (1)

$= \underbrace{\epsilon_{kij} W_k (\partial_i V_j)}_{} - \underbrace{\epsilon_{jik} V_j (\partial_i W_k)}_{}$

(0,5) $\vec{W}\cdot(\vec{\nabla}\times\vec{V}) = 0 \qquad \vec{V}\cdot(\vec{\nabla}\times\vec{W})$ (0,5)

$= -\vec{r}\cdot(\vec{\nabla}\times\vec{W})$, mit $\vec{V}(\vec{r}) = \vec{r}$, $\vec{\nabla}\times\vec{r} = 0$

b) $j^\mu A_\mu \to j^\mu(A_\mu + \partial_\mu \Lambda) = j^\mu A_\mu + j^\mu(\partial_\mu \Lambda)$ (1)

$= j^\mu A_\mu + \partial_\mu(j^\mu \Lambda) = j^\mu A_\mu + \partial_\mu K^\mu$,

(1)

wg. $\underline{\partial_\mu j^\mu = 0}$. $\boxed{K^\mu = j^\mu \Lambda}$

c) $\varphi_{\text{Punkt}} = \frac{Q}{r}$ (0,5), $\frac{p}{r^2} < \varphi_{Dip.} < \frac{p}{r^2}$ (0,5) $\left(\text{wg. } \varphi_{Dip.} = \frac{\vec{p}\cdot\vec{r}}{r^3}\right)$

$\Rightarrow \delta\varphi = \frac{\delta\varphi_{Dip}}{\varphi_{\text{Punkt}}}$ (1) $= (p/r^2)/(Q/r) = \frac{p}{Qr} = \frac{3\,\text{cm}}{1\,\text{m}} = \underline{\underline{3\,\%}}$

d) Maxwell-Gl.: $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$, $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E}$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0} - \Delta\vec{E}$ (mit Hinweis (*)) $= -\Delta\vec{E}$ (0,5)

(0,5)

$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times(-\partial_t\vec{B}) = -\frac{1}{c^2}\partial_t^2\vec{E} \Rightarrow \left(\frac{1}{c^2}\partial_t^2\vec{E} - \Delta\vec{E}\right) = 0$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\Delta\vec{B} = \vec{\nabla}\times\left(\frac{1}{c^2}\partial_t\vec{E}\right) = -\frac{1}{c^2}\partial_t^2\vec{B}$ (0,5)

(0,5)

$\Rightarrow \left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\vec{B} = 0$

e) $j^\mu = (c\rho, \vec{j})$ ist 4-Vektor (1)

$\Rightarrow c\rho' = \gamma(c\rho - \beta j_x)$, $j_x' = \gamma(j_x - \beta c\rho)$,

(1)

$j_y' = j_y$, $j_z' = j_z$

(*)

$\epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l E_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\partial_j\partial_l E_m = \partial_i(\partial_m E_m) - \partial_j^2 E_i$

2 $\rho(\vec{r}) = \delta(r-R)\,\sigma_0\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right)$

$$= \delta(r-R)\,\sigma_0\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) = \delta(r-R)\underbrace{\sigma_0\sqrt{\frac{8\pi}{5}}}_{\equiv K}Y_{20}$$

$$= \delta(r-R)\,K\,Y_{20}(\theta) \equiv \delta(r-R)\,\sigma(\theta) \quad (1)$$

a) $\varphi(\vec{r}) = \frac{1}{\cancel{4\pi}\epsilon_0}\int d^3r'\,\delta(r'-R)\,K\,Y_{20}(\theta')$

$$\times \sum_{l,m}\frac{\cancel{4\pi}}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

$$= \frac{1}{5\epsilon_0}\int_0^\infty dr'\,r'^2\,\frac{r_<^2}{r_>^3}\,K\,\delta(r'-R)\,Y_{20}(\theta) \quad (1)$$

Hier wurde die Orthogonalität d. Y_{lm} verwendet.

$\underline{r<R:}$ $\varphi_<(\vec{r}) = \frac{1}{5\epsilon_0}\frac{r^2}{R}K\,Y_{20}(\theta) \equiv f(\theta)\frac{r^2}{R}$ (1)

$\underline{r>R:}$ $\varphi_>(\vec{r}) = \frac{1}{5\epsilon_0}\frac{R^4}{r^3}K\,Y_{20}(\theta) \equiv f(\theta)\frac{R^4}{r^3}$ (1)

b) $\vec{E}_<(\vec{r}) = -\vec{\nabla}\varphi_<(\vec{r}) = -\frac{2}{\cancel{\epsilon_0}}\frac{r}{R}f(\theta)\,\vec{e}_r - \frac{1}{\cancel{\epsilon_0}}\frac{r}{R}\partial_\theta f(\theta)\,\vec{e}_\theta$ (2)

$\vec{E}_>(\vec{r}) = \frac{3}{\cancel{\epsilon_0}}\frac{R^4}{r^4}f(\theta)\,\vec{e}_r - \frac{1}{\cancel{\epsilon_0}}\frac{R^4}{r^4}\partial_\theta f(\theta)\,\vec{e}_\theta$ (2)

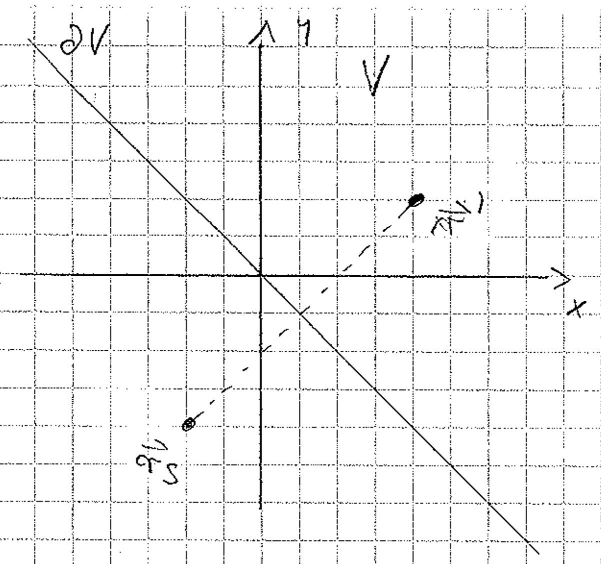
c) Normalkomponente: $(\vec{e}_r\cdot\vec{e}_\theta = 0)$

$\vec{e}_r\cdot(\vec{E}_< - \vec{E}_>)\big|_{r=R} = -\frac{5}{\cancel{\epsilon_0}}f(\theta) = -\frac{\sigma(\theta)}{\epsilon_0}$ (1)

Tangentialkomponente:

$\vec{e}_r\times(\vec{E}_< - \vec{E}_>)\big|_{r=R} = 0$ (1)

3) Allgemein:



Ladung q' bei $\vec{r}\,'$,

Spiegelladung $q_S = -q'$ bei $\vec{r}_S$

a) Sei $q_S = -q'$ ①

$$\Rightarrow G_D(\vec{r} - \vec{r}\,') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z'-z)^2}} - \frac{1}{\sqrt{(x-x_S)^2 + (y-y_S)^2 + (z-z_S)^2}}\right] \quad ①$$

Sei $G_D(\vec{r} - \vec{r}\,') = 0$ für $x = -y$ ①

$$\Rightarrow \sqrt{(-y-x')^2 + (y-y')^2 + (z-z')^2} = \sqrt{(-y-x_S)^2 + (y-y_S)^2 + (z-z_S)^2}$$

$$\Rightarrow \boxed{x_S = -y', \quad y_S = -x', \quad z_S = z'} \quad ①$$

b) Sei Ladung q bei $\vec{r}\,' = (a, a, 0)^T$

$\Rightarrow$ Spiegelladung $-q$ bei $\vec{r}_S = (-a, -a, 0)^T$ ②

Man erhält: ②

$$\varphi_b(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{((x-a)^2 + (y-a)^2 + z^2)^{1/2}} - \frac{1}{((x+a)^2 + (y+a)^2 + z^2)^{1/2}}\right]$$

c) i) $G_D(\vec{r}, \vec{r}\,')$ unverändert ②

ii) $\varphi_c(\vec{r}) = \varphi_b(\vec{r}) + \varphi_0$ ②

4| $\vec{A} = \frac{1}{2}(\vec{h} \times \vec{r}) = \frac{1}{2}\Big[\vec{e}_x (h_y z - h_z y) + \vec{e}_y (h_z x - h_x z)$

$+ \vec{e}_z (h_x y - h_y x)\Big]$

a) $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{2}\left[\vec{e}_x (2h_x) + \vec{e}_y (2h_y) + \vec{e}_z (2h_z)\right]$

$= \boxed{\vec{h}}$ (2) (*)

b) $h_x = h_y = 0, \; h = h_z \Rightarrow \vec{A} = \frac{1}{2}\left[(-h \cdot y)\vec{e}_x + h x \vec{e}_y\right]$ (2)

c) $\vec{A}' = -h y \vec{e}_x$ (1), $\vec{B}' = \vec{\nabla} \times \vec{A}' = -\vec{e}_z \partial_y A'_x = h \vec{e}_z = \vec{B}$ (1)

d) $\vec{A}' = \vec{A} - \vec{\nabla}\Lambda(\vec{r}) \Rightarrow \vec{\nabla}\Lambda(\vec{r}) = \vec{A} - \vec{A}'$ (1)

$= \frac{1}{2} h y \vec{e}_x + \frac{1}{2} h x \vec{e}_y \Rightarrow \Lambda(\vec{r}) = \frac{1}{2} h (x y)$ (1)

(*) <u>Alternativ:</u>

$[\vec{\nabla} \times (\vec{h} \times \vec{r})]_i = \epsilon_{ijk} \partial_j \epsilon_{klm} h_l x_m = \epsilon_{ijk} \epsilon_{klm} h_l \delta_{jm}$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) h_l \delta_{jm}$

$= 3h_i - \delta_{ij} h_j = 3h_i - h_i = 2h_i$

$\Rightarrow \vec{\nabla} \times \frac{1}{2}(\vec{h} \times \vec{r}) = \vec{h}$