Classical Theoretical Physics III

Prof. Dr. Kirill Melnikov – Winter Semester 2025/2026

Date: 05.03.2026, Duration: 120 mins

Name, Surname: [redacted]

Student registration number:

The total number of points is 120.
However, **to get the highest grade, it is sufficient to collect 100 points.**
You can skip questions and solve them in the order you prefer.

Exam Rules

- Please turn over this sheet only after the start of the exam has been announced.
- Please write your name on each sheet of paper.

  Use only the paper provided. If you need additional paper, please inform the supervisory staff.
- Work on each new problem on a separate sheet.

  Please only use black or blue pens, do not use pencils or red or green pens.

  At the end of the examination paper you will find a short collection of formulas. The use of any other aids is not permitted. Please, do not keep your phone, or any other electronic device, on your table.

  Anyone who needs to go to the restroom must hand in the examination paper, solutions and the smartphone to the supervisory staff and will receive them back afterwards.
- You may hand in your examination up to 20 minutes before the end of the examination time. If you finish within the last 20 minutes, please remain seated at your desk until the end of the examination.

Summary of Points

Exercise 1	**Exercise 2**	**Exercise 3**	**Total**
[/35 Points]	[/40 Points]	[/45 Points]	[/100 Points]

Exercise 1 – Spherical shell [35 Points]

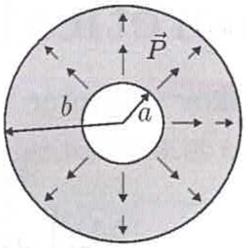

Figure 1.1: Spherical shell with inner and outer radii $a < b$.

A spherical shell with inner radius a and outer radius $b > a$ is made of a material with frozen-in polarization

$$\vec{P}(r) = \frac{k}{r}\theta(r-a)\theta(b-r)\,\vec{e}_r\,, \qquad k > 0\,. \tag{1.1}$$

A) **[5 Points]** Write the continuity conditions for the fields $\vec{E}$ and $\vec{D}$.

B) **[5 Points]** Find the electric field $\vec{E}$ in the whole space (for $r < a$, $a \le r \le b$, and $b < r$).

C) **[5 Points]** Find the electric potential φ in the whole space. Use the boundary condition $\lim_{r\to\infty}\varphi(\vec{r}) = 0$.

D) **[2.5 Points]** Sketch the function $\varphi(r)$.

E) **[5 Points]** Write down how the densities of free and bound charges ρ_f and ρ_b are related to the polarization $\vec{P}$.

F) **[7.5 Points]** Calculate the densities ρ_f and ρ_b of free and bound charges in the whole space.

G) **[5 Points]** Discuss which volumes and surfaces are charged positively or negatively. What is the total charge of the spherical shell?

Exercise 2 – Tenuous plasma and waves **[40 Points]**

A "tenuous plasma" consists of *non-interacting* free electric charges of mass m and charge $-e$ in a background of ions with charges $q = +e$ (the ions are not moving). There are N positive and N negative charges per unit volume, where N is constant. Assume that the density is uniform and that the plasma is *neutral overall.* A monochromatic electromagnetic plane wave with frequency ω and wave number k propagates in vacuum and is incident on the plasma. Throughout this exercise, we work in the non-relativistic limit, that is, the absolute value of the velocity $\vec{v}$ of the free charges is much smaller than the speed of light.

A) **[5 Points]** Write the equation of motion for a single charge at position $\vec{r}$. Explain why the contribution from the *magnetic* field of the wave can be neglected in the equation of motion.

B) **[7.5 Points]** Using point A) and the Ohm's law,

$$\vec{j} = \sigma \vec{E}\,, \tag{2.1}$$

where $\vec{j}$ is the current density, show that the conductivity σ of the plasma as a function of ω is given by

$$\sigma = \mathrm{i}\,\frac{Ne^2}{m\omega}\,. \tag{2.2}$$

[***Hint:*** *The general solution of the equation of motion found in point A) for an electron in* $\vec{r}_0$ *is such that* $\vec{r}(t) = \vec{r}_0 + \delta\vec{r}(t)$, *where* $\delta\vec{r}(t) \perp \vec{k}$.]

C) **[10 Points]** Starting from Maxwell's equations, derive the wave equation for a wave propagating in the plasma. The currents $\vec{j}$ in the plasma and the electric field $\vec{E}$ of the wave are related by Eq. (2.1). Use the wave equation to find the relation between k and ω.

D) **[5 Points]** Find the index of refraction n as a function of ω. The plasma frequency is defined by

$$\omega_{\mathrm{p}}^2 = \frac{Ne^2}{m\epsilon_0}\,. \tag{2.3}$$

Show that n becomes imaginary if $\omega < \omega_{\mathrm{p}}$. What are the frequencies ω of waves that can propagate over long distances in the plasma?

E) **[12.5 Points]** Suppose that the plasma is placed in an external magnetic field $\vec{B}_0 = B_0\,\vec{e}_z$. Consider circularly polarized plane waves travelling inside the plasma parallel to $\vec{B}_0$, i.e.,

$$\vec{E}_{\pm}(t,\vec{r}) = E_0 e^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\left[\frac{\vec{e}_x \pm \mathrm{i}\vec{e}_y}{\sqrt{2}}\right] = E_0 e^{\mathrm{i}(kz-\omega t)}\vec{e}_{\pm}\,. \tag{2.4}$$

Write the equation of motion for this case, accounting for the external magnetic field $\vec{B}_0$. Show that the index of refraction is different for $\vec{e}_+$ and $\vec{e}_-$ polarizations (assume that the magnetic field of the travelling waves is negligible compared to external field $\vec{B}_0$).

Exercise 3 – Particle inside a uniformly-charged ball **[45 Points]**

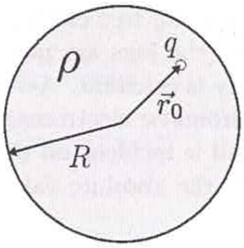

Figure 3.1: Test point particle of charge q inside a charged ball with uniform charge density ρ.

A test point particle of charge q and mass m is moving inside a uniformly-charged ball with charge density ρ and radius R. The particle and the ball are oppositely charged, i.e., $q\rho < 0$. At $t = 0$, the particle is at rest at distance $r_0 < R$ from the center of the ball.

A) **[5 Points]** Find the electric field $\vec{E}$ inside the ball. Neglect the field generated by the test charge.

B) **[5 Points]** Write down the equations of motion for the particle. Assume that the only force acting on the test charge is due to the electric field generated by the ball. Assume that the particle velocity v is much smaller than the speed of light.

C) **[5 Points]** Solve the equations of motion with the initial conditions $r(t = 0) = r_0$ and $v(t = 0) = 0$.

D) **[5 Points]** Find the electric and magnetic fields $\vec{E}$ and $\vec{B}$ in the waves radiated by the particle, far away from the ball.

E) **[5 Points]** Calculate the Poynting vector $\vec{S}$ for the waves radiated by the particle.

F) **[5 Points]** What is the radiated power $\mathcal{P}(t)$?

G) **[5 Points]** Determine the energy the particle loses by the time it reaches the center of the ball, starting from the initial conditions given in point C).

H) **[10 Points]** Estimate the time required for the test charge to lose half of its energy due to radiation. Choose the additive constant in the potential energy of the test charge such that its potential energy is zero in the center of the ball.

Collection of formulae

Cylindrical coordinates:

$$
\begin{aligned}
(x, y, z) &= (\rho\cos\phi, \rho\sin\phi, z)\,,\\
\vec{\nabla} f &= \frac{\partial f}{\partial \rho}\,\vec{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\vec{e}_\phi + \frac{\partial f}{\partial z}\,\vec{e}_z\,,\\
\vec{\nabla}^2 f &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}\,,\\
\vec{\nabla}\cdot\vec{A} &= \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\,,\\
\vec{\nabla}\times\vec{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{e}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\vec{e}_z\,.
\end{aligned}
$$

Spherical coordinates:

$$
\begin{aligned}
(x, y, z) &= r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,,\\
\vec{\nabla} f &= \frac{\partial f}{\partial r}\,\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\vec{e}_\phi\,,\\
\vec{\nabla}^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}\,,\\
\vec{\nabla}\cdot\vec{A} &= \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}\,,\\
\vec{\nabla}\times\vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\rho)}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{e}_\phi\,.
\end{aligned}
$$

Fresnel's formulae:

$$
r_\mathrm{s} = \frac{n_\mathrm{I}\cos\theta_\mathrm{I} - n_\mathrm{T}\cos\theta_\mathrm{T}}{n_\mathrm{I}\cos\theta_\mathrm{I} + n_\mathrm{T}\cos\theta_\mathrm{T}}\,, \qquad t_\mathrm{s} = \frac{2n_\mathrm{I}\cos\theta_\mathrm{I}}{n_\mathrm{I}\cos\theta_\mathrm{I} + n_\mathrm{T}\cos\theta_\mathrm{T}}\,,
$$

$$
r_\mathrm{p} = \frac{n_\mathrm{T}\cos\theta_\mathrm{I} - n_\mathrm{I}\cos\theta_\mathrm{T}}{n_\mathrm{T}\cos\theta_\mathrm{I} + n_\mathrm{I}\cos\theta_\mathrm{T}}\,, \qquad t_\mathrm{p} = \frac{2n_\mathrm{I}\cos\theta_\mathrm{I}}{n_\mathrm{T}\cos\theta_\mathrm{I} + n_\mathrm{I}\cos\theta_\mathrm{T}}\,.
$$

Potentials of a dipole far away from the source (τ is the retarded time):

$$
\phi = \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{\vec{n}}{c}\cdot\frac{\partial\vec{p}(\tau)}{\partial\tau}\right]\,, \qquad \vec{A} = \frac{1}{4\pi\epsilon_0 r}\frac{1}{c^2}\frac{\partial\vec{p}(\tau)}{\partial\tau}\,,
$$

Power radiated by a relativistic point charge:

$$
\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}^2\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c}\frac{\left|\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right|^2}{(1-\vec{\beta}\cdot\vec{n})^5}\,, \qquad \mathcal{P} = \frac{q^2\gamma^6}{6\pi\epsilon_0 c}\left[\dot{\beta}^2 - \left|\dot{\vec{\beta}}\times\vec{\beta}\right|^2\right]\,.
$$