Klausur 1 - Lösungen Thema C - Elektrodynamik WS 99/2000

Aufgabe 1

a) $\boxed{D}$ $\vec{\nabla} \times (\hat{z} \times x\hat{x}) = \vec{\nabla} \times (x\hat{y}) = \boxed{\hat{z}}$

) $\boxed{C}$ $\vec{\nabla} \times (\hat{\phi} \times r\hat{r}) = \vec{\nabla} \times (r\hat{\theta}) = \frac{1}{r}\frac{\partial}{\partial r}(r^2)\hat{\phi} = \boxed{2\hat{\phi}}$

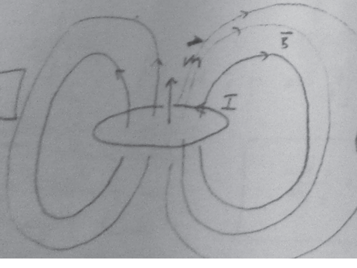
i) $\boxed{B}$ $\sigma = \varepsilon_0 \vec{E}\cdot\hat{n} \overset{\text{Gauß}}{=} \varepsilon_0 \left(\frac{q_a + q_b}{4\pi R^2}\frac{1}{\varepsilon_0}\right) = \boxed{\frac{q_a + q_b}{4\pi R^2}}$

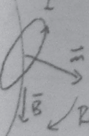
ii) $\boxed{B}$ Das Feld der anderen Ladungen außer q_a dringt nicht in den Hohlraum um q_a ein, folglich ist die Kraft auf q_a gleich $\boxed{\text{Null}}$.

) $\boxed{C}$ $\vec{F}_{mag} = I\int(d\vec{l} \times \vec{B}) = IaB\hat{z}$ wenn Strom im Uhrzeigersinn fließt. $F_{mag} = mg \Rightarrow \boxed{I = \frac{mg}{Ba}}$

$\boxed{D}$ $C = K\, C_{Luft}$

$\boxed{B}$





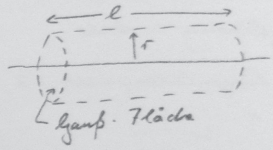
Gleichgewicht:

$\vec{m} \parallel \vec{B} \Rightarrow \boxed{m \parallel -\hat{z}}$

Richtung der Kraft, die $\vec{m}$ durch $\vec{B}$ empfindet

$\vec{N} = \vec{m} \times \vec{B} =$

(2) Langer Draht

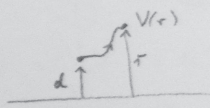


(2a) $\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{eingeschlossen}$

$E(r)\, 2\pi r l = \frac{1}{\varepsilon_0}\lambda l$

$\vec{E}(\vec{r}) = E(r)\,\hat{r}$

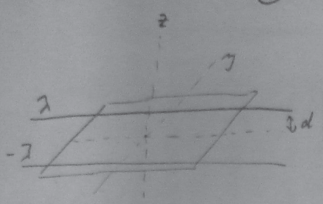
$$\boxed{\vec{E}(r) = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{r}}$$



(2b) $V(r) = -\int_d^r \vec{E}\cdot d\vec{l} = -\int_d^r dr' \left(\frac{\lambda}{2\pi\varepsilon_0 r'}\right)$

$$\boxed{V(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r/d)}$$

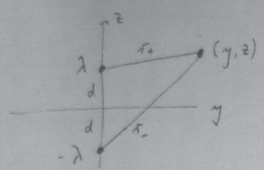
Spiegelladung : Draht mit Längenladungsdichte $-\lambda$ bei $-d$.



Potential :

$$V(y,z) = V_+ + V_-$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\ln(r_+/d) + \frac{\lambda}{2\pi\varepsilon_0}\ln(r_-/d)$$



$$\boxed{V(y,z) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{(z+a)^2+y^2}{(z-a)^2+y^2}\right]}$$ (für $y>0$)

Induzierte Ladungsdichte :

$$\sigma(y) = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

$$= -\frac{\lambda}{2\pi}\left[\frac{2(z+a)}{(z+a)^2+y^2} - \frac{2(z-a)}{(z-a)^2+y^2}\right]_{z=0}$$

$$\boxed{\sigma(y) = -\frac{2\lambda a}{\pi(a^2+y^2)}}$$

(3) Potential auf der Oberfläche einer Hohlkugel

$V(R,\theta) = V_0(\theta)$

(3a)

$$V_0(\theta) = k\cos(2\theta)$$

$$= k(\cos^2\theta - \sin^2\theta)$$

$$= k(2x^2-1) \qquad (x=\cos\theta)$$

$$= k\left\{\frac{4}{3}\left[\frac{1}{2}(3x^2-1)\right] - \frac{1}{3}\right\}$$

$$= k\left\{\frac{4}{3}P_2(x) - \frac{1}{3}P_0(x)\right\} \qquad (*)$$

(3b) Innerhalb der Kugel:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell\, r^\ell\, P_\ell(\cos\theta) \qquad (x=\cos\theta)$$

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) \equiv V_0(\theta) \qquad (*)$$

$$\int_{-1}^{1} dx\, P_\ell(x)\, V(R,x) = \sum_{\ell'=0}^{\infty} A_{\ell'} R^{\ell'} \underbrace{\int_{-1}^{1} dx\, P_{\ell'}(x) P_\ell(x)}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_{-1}^{1} dx\, P_\ell(x)\, V_0(x)$$

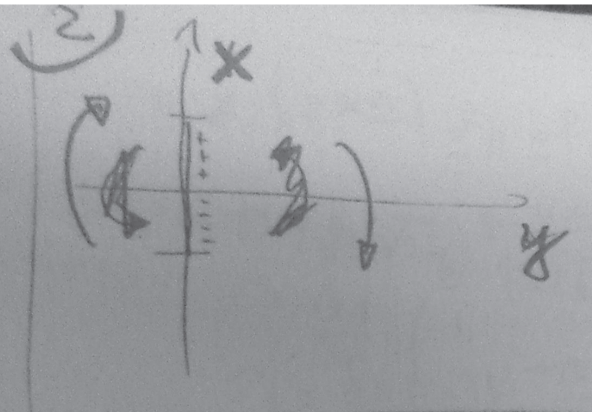
$$= \frac{2\ell+1}{2R^\ell}\left\{\frac{2}{2\ell+1}\, k\left[\frac{4}{3}\delta_{\ell 2} - \frac{1}{3}\delta_{\ell 0}\right]\right\}$$

nicht unbedingt nötig !

$$\boxed{A_0 = -\frac{k}{3}, \qquad A_2 = \frac{4k}{3R^2}}$$

Dieses hätte man per Inspektion aus (*) nur direkt folgen können

2)



Bem.: ausführliche Lösung mit Herleitung der Formeln

$$\rho(\underline{x},t)\Big|_{t=0} = \frac{Q}{d}\,\delta(y)\,\delta(z)\cdot\begin{cases}1, & 0<x\le\frac{d}{2}\\ -1, & -\frac{d}{2}\le x\le 0\end{cases}$$

(a)

$$\underline{p}\Big|_{t=0} = \int d^3\underline{x}'\,\rho(\underline{x}')\cdot\underline{x}' \qquad \boxed{1P}$$

$$= \int dx\,dy\,dz\,\frac{Q}{d}\,\delta(y)\,\delta(z)\cdot\begin{Bmatrix}1\\-1\end{Bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$= \frac{Q}{d}\left(\int_0^{d/2}dx - \int_{-d/2}^{0}dx\right)\begin{pmatrix}x\\0\\0\end{pmatrix}.$$

$$= \frac{Q}{2d}\underline{e}_x\left(x^2\Big|_0^{d/2} - x^2\Big|_{-d/2}^{0}\right) = \frac{Q}{2d}\left(\frac{d^2}{4}-\left(-\frac{d^2}{4}\right)\right)\underline{e}_x$$

$$= \frac{Q}{2d}\,\frac{d^2}{2}\,\underline{e}_x = \underline{\frac{Qd}{4}\,\underline{e}_x}. \qquad \boxed{1P}$$

Dipolmoment rotiert mit dem Drahtstück in der x-y-Ebene.

$$\underline{p}(t) = \left(\underline{e}_x\cos(\omega t)+\underline{e}_y\sin(\omega t)\right)\frac{Qd}{4} \qquad \underline{e}_+ = \frac{1}{\sqrt{2}}(1,i,0)$$

$$= \mathrm{Re}\left\{\frac{Qd}{4}\begin{pmatrix}1\\i\\0\end{pmatrix}e^{-i\omega t}\right\} \Rightarrow \boxed{\underline{p} = \left(\frac{Qd}{2\sqrt{2}}\right)\underline{e}_+}$$

$\boxed{1P}$

(b) s. Jackson, 3. Aufl., S. 407–411

$$\rho(\underline{x},t)=\rho(\underline{x})\,e^{-i\omega t},\quad \underline{J}(\underline{x},t)=\underline{J}(\underline{x})\,e^{-i\omega t} \quad \text{Fouriertransf.}$$

$$\underline{A}(\underline{x},t) = \frac{\mu_0}{4\pi}\int d^3\underline{x}'\int dt'\,\frac{\underline{J}(\underline{x}',t')}{|\underline{x}-\underline{x}'|}\,\delta\!\left(t'+\frac{|\underline{x}-\underline{x}'|}{c}-t\right)$$

$$= \frac{\mu_0}{4\pi}\int d^3\underline{x}'\int dt'\,\frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\,e^{-i\omega t'}\,\delta\!\left(t'+\frac{|\underline{x}-\underline{x}'|}{c}-t\right)$$

$$= \left(\frac{\mu_0}{4\pi}\int d^3\underline{x}'\,\frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\,e^{+i\frac{|\underline{x}-\underline{x}'|}{c}}\right)\cdot e^{-i\omega t}$$

Fern-Strahlungs-Zone: $\quad d \ll \lambda \ll r$

$$|\underline{x}-\underline{x}'| \approx r - \underline{n}\cdot\underline{x}' \quad , \quad \underline{n}^2 = 1 , \quad \underline{n} \parallel \underline{x} \qquad k = \omega/c$$

$$\underline{A}(\underline{x}) \xrightarrow{kr \to \infty} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\underline{x}'\, \underline{J}(\underline{x}')\, e^{-ik\underline{n}\underline{x}'}$$

$d \ll \lambda$

$$\longrightarrow \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{m=0}^{\infty}\int d^3\underline{x}'\, \frac{(-ik)^m}{m!}\,\underline{J}(\underline{x}')\,(\underline{n}\cdot\underline{x}')^m$$

$$\xrightarrow{m=0} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\underline{x}'\, \underline{J}(\underline{x}') \qquad \boxed{1P}$$

$$\int \underline{J}\, d^3\underline{x}' = -\int \underline{x}'(\underline{\nabla}'\underline{J})\, d^3\underline{x}' \underset{\text{Kontinuitätsgl.}}{=} -i\omega\int \underline{x}'\rho(\underline{x}')\, d^3\underline{x}'$$

$$\Rightarrow \quad \underline{A}(\underline{x}) = \frac{-i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\underbrace{\int d^3\underline{x}'\, \underline{x}'\rho(\underline{x}')}_{\underline{p}} \qquad \boxed{1P}$$

$$\Rightarrow \quad \boxed{\underline{A}(\underline{x},t) = \mathrm{Re}\left[\frac{-i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\,\underline{p}\, e^{-i\omega t}\right]} \qquad \boxed{1P}$$

$$\underline{B} = \underline{\nabla}\times\underline{A}(\underline{x},t) = \mathrm{Re}\left[\frac{-i\mu_0\omega}{4\pi}\underbrace{\underline{\nabla}\times\left(\frac{e^{ikr}}{r}\underline{p}\right)}\, e^{-i\omega t}\right]$$

$$\underline{\nabla}\times(\psi\underline{a}) = \underline{\nabla}\psi\times\underline{a} + \cancel{\psi(\underline{\nabla}\times\underline{a})} \qquad \left(\underline{\nabla}\frac{e^{ikr}}{r}\right)\times\underline{p} .$$

$$\underline{\nabla}\frac{e^{ikr}}{r} = \frac{\partial r}{\partial \underline{r}}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) = \frac{\underline{r}}{r}\left(\frac{ik}{r} - \frac{1}{r^2}\right)e^{ikr}$$

$$= \frac{ik\underline{r}}{r^2}\left(1 - \cancel{\frac{1}{ikr}}\right)e^{ikr} \qquad \frac{\underline{r}}{r} = \underline{n}$$

vernachlässige das!

$$\Rightarrow \quad \boxed{\underline{B}(\underline{x},t) = \mathrm{Re}\left[\frac{\mu_0 c k^2}{4\pi}(\underline{n}\times\underline{p})\frac{e^{ikr}}{r}e^{-i\omega t}\right]} \qquad \boxed{2P}$$

$$\underline{n} \times (\underline{p} \times \underline{n}) = \underline{p}\, n^2 - \underline{n}(\underline{n}\underline{p}) \qquad \underline{n} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$

$$= \underline{p} - \underline{n} \frac{Qd}{4} \sin\theta\, e^{i\phi}$$

$$= \frac{Qd}{4} \begin{pmatrix} 1 - \sin^2\theta \cos\phi\, e^{i\phi} \\ i - \sin^2\theta \sin\phi\, e^{i\phi} \\ -\cos\theta \sin\theta\, e^{i\phi} \end{pmatrix},$$

$$\Rightarrow \quad |\underline{n} \times (\underline{p} \times \underline{n})|^2 = \frac{Q^2 d^2}{16} \Big\{ |1 - \sin^2\theta \cos\phi\, e^{i\phi}|^2 + |i - \sin^2\theta \sin\phi\, e^{i\phi}|^2 + \cos^2\theta \sin^2\theta \Big\}.$$

$$\{\ldots\} = 1 + \underline{\sin^4\theta \cos^2\phi} \underbrace{-2\sin^2\theta \cos^2\phi} + 1 + \underline{\sin^4\theta \sin^2\phi} + \cos^2\theta \sin^2\theta \underbrace{-2\sin^2\theta \sin^2\phi} + \cos^2\theta \sin^2\theta.$$

$$= 2 + \sin^4\theta - 2\sin^2\theta + \cos^2\theta \sin^2\theta = 1 + \cos^2\theta.$$

$$= 2\cos^2\theta + \sin^2\theta$$

$$\Rightarrow \quad \boxed{\frac{dP}{d\Omega} = \frac{c}{32\pi^2 \varepsilon_0} k^4 \frac{Q^2 d^2}{16} (1 + \cos^2\theta)} \qquad \boxed{2P}$$

$$\boxed{\frac{dP}{d\cos\theta} = \frac{Q^2 d^2 c}{64 (4\pi\varepsilon_0)} k^4 (1 + \cos^2\theta)}$$

$$\boxed{P = \int_{-1}^{1} \frac{dP}{d\cos\theta} d(\cos\theta) = \frac{Q^2 d^2 c}{24 (4\pi\varepsilon_0)} k^4} \qquad \boxed{1P}$$

(e) Natzone

$$d \ll r \ll \lambda. \quad \Rightarrow (kr) \ll 1.$$

$$\Rightarrow e^{ikr} \longrightarrow 1.$$

$$\Rightarrow \quad \underline{A}(\underline{x}) = -\frac{i\mu_0 \omega}{4\pi r} \underline{p}$$

$\underline{E}(\underline{x}, t) = \mathrm{Re}\left[\underline{E}(\underline{x}) e^{-i\omega t}\right]$

$\Rightarrow \underline{E}(\underline{x}) = \underline{E}(\underline{x}, \theta).$

$$\boxed{\underline{E}(\underline{x}) = \frac{ic^2}{\omega} \nabla \times (\nabla \times \underline{A})}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{3\underline{n}(\underline{n}\underline{p}) - \underline{p}}{r^3}$$

4

Dipolfeld

$$\frac{\partial \underline{E}}{c^2 \partial t} = \underline{\nabla} \times \underline{B} \qquad \text{(außerhalb des Gebietes mit dem Strom)}$$

$$\Rightarrow -\frac{i\omega}{c}\underline{E} = \underline{\nabla} \times \underline{B} \quad \Rightarrow \quad \boxed{\underline{E} = \frac{ic^2}{\omega}\underline{\nabla} \times \underline{B} = \frac{ic}{k}\underline{\nabla} \times \underline{B}} \quad \boxed{1P}$$

$$\underline{\nabla} \times \left(\frac{e^{ikr}}{r}(\underline{n} \times \underline{p})\right)$$

Ableitungen von $\underline{n}$ bringt höhere Potenzen von $\frac{1}{r} \longrightarrow$ vernachlässigt

$$\approx \left(\underline{\nabla}\frac{e^{ikr}}{r}\right) \times (\underline{n} \times \underline{p}) \approx \frac{ik}{r}e^{ikr} \; \underline{n} \times (\underline{n} \times \underline{p}).$$

$$\Rightarrow \text{Vorfaktor } \frac{\mu_0 c^2 k^2}{4\pi} = \frac{k^2}{4\pi\varepsilon_0} \qquad \boxed{2P}$$

$$\Rightarrow \boxed{\underline{E}(x,t) = \mathrm{Re}\left[\frac{1}{4\pi\varepsilon_0} k^2 (\underline{n} \times \underline{p}) \times \underline{n} \; \frac{e^{ikr}}{r} e^{-i\omega t}\right]}$$

(c) Polarisation s.u. entlang des Vektors $(\underline{n} \times \underline{p}) \times \underline{n}$ $\boxed{1P}$

$$\underline{n} = \underline{e}_z \qquad (\underline{e}_z \times \underline{p}) \times \underline{e}_z = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \times \underline{e}_z = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \parallel \underline{p}$$

$\Rightarrow$ zirkular polarisierte Welle. $\boxed{1P}$

$$\underline{n} = \underline{e}_x \qquad (\underline{e}_x \times \underline{p}) \times \underline{e}_x = \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} \times \underline{e}_x = i\,\underline{e}_y$$

$\Rightarrow$ linear polarisierte Welle. $\boxed{1P}$

(d)
$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0}\mathrm{Re}\left[r^2 \underline{n}(\underline{E} \times \underline{B}^*)\right] \qquad \boxed{1P}$$

Vorfaktoren:
$$\frac{k^2}{4\pi\varepsilon_0}\frac{\cancel{e^{ikr}}}{r}\cancel{e^{-i\omega t}} \cdot \frac{\mu_0 c k^2}{4\pi}\frac{\cancel{e^{-ikr}}}{r}\cancel{e^{i\omega t}}$$

$$= \frac{\mu_0 c k^4}{16\pi^2 \varepsilon_0 r^2} \Rightarrow$$

$$\frac{dP}{d\Omega} = \frac{\cancel{\mu_0} c k^4 \cancel{r^2}}{32\cancel{\mu_0}\pi^2\varepsilon_0 \cancel{r^2}} \mathrm{Re}\Big[\underbrace{\underline{n} \cdot \big(((\underline{n} \times \underline{p}) \times \underline{n}) \times (\underline{n} \times \underline{p}^*)\big)}\Big]$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a})$$
$$= ((\underline{n} \times \underline{p}) \times \underline{n}) \cdot ((\underline{n} \times \underline{p}^*) \times \underline{n})$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c}{32\pi^2\varepsilon_0} k^4 \left|(\underline{n} \times \underline{p}) \times \underline{n}\right|^2} \qquad \boxed{2P}$$

3

Außerhalb der Kugel:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$V(R,\theta) = \sum_{\ell=0}^{\infty} B_\ell \frac{1}{R^{\ell+1}} P_\ell(x) \equiv V_0(\theta)$$

$$\Rightarrow \boxed{B_0 = -\frac{1}{3} R k \quad , \quad B_2 = \frac{4}{3} R^3 k}$$

(durch direkten Vergleich mit (*), oder via

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_{-1}^{1} dx \, P_\ell(x) \, V_0(x) \,)$$

$$\boxed{V(r) = k \begin{cases} \frac{4}{3}\left(\frac{r}{R}\right)^2 P_2(x) - \frac{1}{3} P_0(x) & \text{für } r \leq R \\ \frac{4}{3}\left(\frac{R}{r}\right)^3 P_2(x) - \frac{1}{3} P_0(x) & \text{für } r \geq R \end{cases}}$$

Flächenladungsdichte:

$$\sigma(\theta) = -\varepsilon_0 \left(\frac{\partial V^{\text{außen}}}{\partial r} - \frac{\partial V^{\text{innen}}}{\partial r} \right)\Bigg|_{r=R}$$

$$= -\varepsilon_0 k \frac{4}{3} P_2(\cos\theta) \left[2\left(\frac{r}{R^2}\right) - (-3)\left(\frac{R^3}{r^4}\right) \right]_{r=R}$$

$$\boxed{\sigma(\theta) = -\frac{20}{3} \frac{\varepsilon_0 k}{R} P_2(\cos\theta)}$$

(4) $\vec{B}$ und $\vec{A}$ innerhalb eines stromtragenden Leiters

(4a)

Stromdichte: $J = \frac{I}{\pi R^2}$

$$\vec{B} = B(r)\,\hat{\phi}$$

Ampère: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 J \pi r^2$

$$B \, 2\pi r = \mu_0 I \left(\frac{r}{R}\right)^2$$

$$\boxed{\vec{B} = \frac{\mu_0 I r}{2\pi R^2} \hat{\phi}}$$

(4b)

Ansatz: $\vec{A}(\vec{r}) = -A(r)\,\hat{z}$

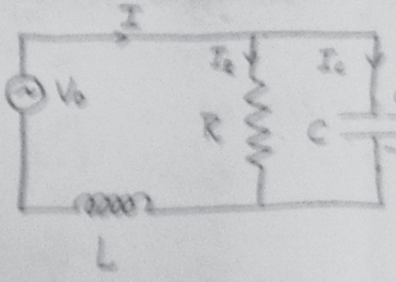
$$\vec{\nabla} \times \vec{A} \overset{(?)}{=} \vec{B} \qquad \text{(in Zylinderkoordinaten)}$$

$$+\frac{\partial A(r)}{\partial r} \hat{\phi} = \vec{B}$$

$$\Rightarrow A(r) = \frac{\mu_0 I r^2}{4\pi R^2}$$

$$\boxed{\vec{A}(\vec{r}) = -\frac{\mu_0 I r^2}{4\pi R^2} \hat{z}}$$

(5) RCL - Stromkreis



(5a) $V_R = V_C \Rightarrow I_R R = \frac{Q}{C}$

$$I = I_R + I_C = \frac{Q}{RC} + \frac{dQ}{dt} \qquad \otimes$$

$$V_0 = V_L + V_C = L\frac{dI}{dt} + \frac{Q}{C} = \boxed{L\frac{d^2Q}{dt^2} + \frac{L}{RC}\frac{dQ}{dt} + \frac{Q}{C} = V_0}$$

(5b) $V_0(t) = \int \frac{d\omega}{2\pi} \tilde{V}(\omega) e^{-i\omega t}$, analog für $\tilde{Q}(\omega)$ und $\tilde{I}(\omega)$

$\Rightarrow$

$$\tilde{V}(\omega) = \tilde{Q}(\omega)\left[-\omega^2 L - i\omega\frac{L}{RC} + \frac{1}{C}\right]$$

$$\boxed{\tilde{Q}(\omega) = \frac{\tilde{V}(\omega)}{-\omega^2 L - i\omega\frac{L}{RC} + \frac{1}{C}}}$$

(5c) $\otimes$ liefert: $\tilde{I}(\omega) = \tilde{Q}(\omega)\left[\frac{1}{RC} - i\omega\right]$

Impedanz: $Z(\omega) = \tilde{V}(\omega) / \tilde{I}(\omega)$

$$= \frac{\tilde{V}(\omega)}{\tilde{Q}(\omega)\left[\frac{1}{RC} - i\omega\right]}$$

$$= \frac{-\omega^2 L - i\omega\frac{L}{RC} + \frac{1}{C}}{\frac{1}{RC} - i\omega}$$

$$\boxed{Z(\omega) = \frac{-\omega^2 LRC - i\omega L + R}{1 - i\omega RC}}$$

$\left(= -i\omega L + \left[\frac{1}{R} - i\omega C\right]^{-1} \right.$ wie zu erwarten