Lösungen zur 2. Übungsklausur Theorie C
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Lösungen vorgestellt von Tobias Kasprzik

Anmerkung: Einzelne Aufgaben werden weiter ausgeführt als verlangt.

Aufgabe 1

a) $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \qquad A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$

$$F'^{\mu\nu} = \partial^\mu(A^\nu + \partial^\nu\Lambda) - \partial^\nu(A^\mu + \partial^\mu\Lambda) = \underbrace{\partial^\mu A^\nu - \partial^\nu A^\mu}_{F^{\mu\nu}} + \underbrace{\partial^\mu\partial^\nu\Lambda - \partial^\nu\partial^\mu\Lambda}_{=0}$$

b) $\partial_\mu F^{\mu\nu} = \mu_o\, j^\nu\,, \qquad \varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = 0 \qquad$ (Jacobi-Identität)

c) $\mu_o\, j^\nu = \partial_\mu F^{\mu\nu}$

$$= \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu\partial^\mu A^\nu - \partial^\nu \underbrace{(\partial_\mu A^\mu)}_{=0} \qquad \text{(Lorenz-Eichung)}$$

d) In K: $\quad \tilde{F}^{\mu\nu} F_{\mu\nu} = -\frac{4}{c}\,\vec{B}\cdot\vec{E} = 0$

d.h. in K' gilt: $\vec{B}'\cdot\vec{E}' = 0 \quad \Rightarrow \vec{B}' \perp \vec{E}'$ (falls beide Felder nicht verschwinden)

e) $\partial_\mu j^\mu = \partial_t\,\varrho + \vec{\nabla}\cdot\vec{\jmath} = 0 \qquad \left[j^\mu = \begin{pmatrix} c\,\varrho \\ \vec{\jmath} \end{pmatrix},\quad \partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla} \end{pmatrix} \right]$

$$\Rightarrow \frac{\partial}{\partial t} \underbrace{\int \mathrm{d}^3 r\, \varrho(\vec{r},t)}_{\equiv Q} = -\int \mathrm{d}^3 r\, \vec{\nabla}\,\vec{\jmath}(\vec{r},t) \overset{\text{Gauß}}{=} -\int \vec{\jmath}(\vec{r},t)\cdot \mathrm{d}\vec{S} = 0$$

$$\Rightarrow \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

f) $\vec{E}(\vec{r},t) = \vec{E}_o \dfrac{e^{\mathrm{i}(kr-\omega t)}}{r}\,,$ verwende $\triangle\,\psi(r) = \dfrac{1}{r^2}\,\partial_r\big(r^2\,\partial_r\psi(r)\big)$

$$\triangle\vec{E} = \vec{E}_0\,\triangle\left(\frac{e^{\mathrm{i}(kr-\omega t)}}{r}\right) = \frac{E_o\,e^{-\mathrm{i}\omega t}}{r^2}\,\frac{\partial}{\partial r}\left(-e^{+\mathrm{i}kr} + \mathrm{i}kre^{\mathrm{i}kr}\right) = -k^2\vec{E}(\vec{r},t)$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,\vec{E}(\vec{r},t) = -\frac{\omega^2}{c^2}\,\vec{E}(\vec{r},t) \quad\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle\right)\vec{E}(\vec{r},t) = 0 \;\text{ mit } \omega = k\,c$$

Aufgabe 2

a)

$$\varrho(\vec{r}) = \varrho_o\cos\theta,\quad \varrho(\vec{r}) = 0 \text{ für } r > R$$

$$\varrho(\vec{r}) = \sqrt{\frac{4\pi}{3}}\,\varrho_o\,Y_{10}(\theta,\phi)\;\Theta(R-r)$$

$$q_{lm} = \sqrt{\frac{4\pi}{3}}\,\varrho_o\int_0^R \mathrm{d}r\,r^{2+l}\underbrace{\int \mathrm{d}\Omega\,Y_{10}(\theta,\phi)\,Y^*_{lm}(\theta,\phi)}_{\delta_{l1}\delta_{m0}}$$

$$= \sqrt{\frac{4\pi}{3}}\,\varrho_o\int_0^R \mathrm{d}r\,r^3\delta_{l1}\delta_{m0}$$

$$= \sqrt{\frac{\pi}{12}}\,\varrho_o\,R^4\,\delta_{l1}\delta_{m0}$$

$$\Rightarrow q_{10} = \sqrt{\frac{\pi}{12}}\,\varrho_o\,R^4\,,\quad \text{alle anderen } q_{lm} = 0$$

b) Sei allgemein $\varrho(\vec{r}) = 0$ für $r < R$, sei hier $r < R$

$$\varphi(\vec{r}) = \frac{1}{\varepsilon_o}\left[\int_{r'<r}\mathrm{d}^3r' + \underbrace{\int_{r'>r}\mathrm{d}^3r'}_{=0}\right]\sum_l\sum_m \frac{\varrho(\vec{r}\,')}{2l+1}\,\frac{r^l_<}{r^{l+1}_>}\,Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$= \frac{1}{\varepsilon_o}\sum_l\sum_m\frac{1}{2l+1}\underbrace{\int_{r'<r}\mathrm{d}^3r'\,\varrho(\vec{r}\,')\,r'^l\,Y^*_{lm}(\theta',\phi')}_{q_{lm}\text{ für } r\to\infty}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$$= \frac{1}{\varepsilon_o}\sum_l\sum_m\frac{1}{2l+1}q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

Hier: $\varphi(\vec{r}) = \dfrac{1}{3\varepsilon_o}\, q_{10}\, \dfrac{Y_{10}}{r^2} = \dfrac{1}{4\pi\varepsilon_o}\, \dfrac{\varrho_o\, \pi\, R^4}{3}\, \dfrac{r\cos\theta}{r^3}$

Identifiziere mit Dipolpotential $\varphi_{\mathrm{D}} = \dfrac{1}{4\pi\varepsilon_o}\, \dfrac{\vec{p}\cdot\vec{r}}{r^3}$.

$\Rightarrow$ Entspricht dem Potential eines Dipols mit $\vec{p} = \dfrac{\varrho_o \pi\, R^4}{3}\, \vec{e}_z$.

c) $Q = \int \mathrm{d}^3 r\, \varrho(\vec{r}) = 0 \qquad \left(\text{wegen } \int_{-1}^{1} \mathrm{d}\cos\theta\ \cos\theta = 0\right)$

$$\vec{p} = \varrho_o \underbrace{\int_0^R \mathrm{d}r\, r^3}_{\frac{1}{4}R^4} \underbrace{\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\cos\theta\ \cos\theta \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}}_{\frac{4\pi}{3}\vec{e}_z} = \frac{\varrho_o\, \pi\, R^4}{3}\, \vec{e}_z$$

(Durch die ϕ-Integration von 0 bis 2π über sin bzw. cos sind die x- und y-Komponente null. Dies wird auch in Aufgabe 4 mehrmals angewandt.)

Alle weiteren Multipole verschwinden.

Aufgabe 3

a)

$$\vec{A}(\vec{r},t) = f(x-ct)\, \vec{e}_z, \quad \varphi(\vec{r},t) = 0$$

i)

$$f = f(k(x,t)), \quad \frac{\partial f}{\partial k} = f'$$

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}}{\partial t} = c\ f'(x-ct)\, \vec{e}_z$$
$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A} = -\partial_x A_z\, \vec{e}_y = -f'(x-ct)\, \vec{e}_y$$

ii)

$$\partial_\mu A^\mu = 0, \quad \vec{\nabla}\vec{A} = 0, \quad A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$$
$$\text{Soll:} \quad \partial_\mu A'^\mu = 0, \quad \vec{\nabla}\vec{A}' \neq 0$$
$$\Rightarrow\ \partial_\mu \partial^\mu \Lambda = 0, \quad \triangle\Lambda \neq 0$$

Wähle z.B.: $\Lambda(\vec{r},t) = \Lambda_o \sin(x-ct) \qquad \Rightarrow \vec{A}' = \vec{A} - \Lambda_o \cos(x-ct)\, \vec{e}_z$
$\varphi' = -c\cdot\Lambda_o \cos(x-ct)$

Beachte: $\vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\Lambda$, $\varphi \to \varphi' = \varphi + \partial_t \Lambda$ $\qquad \partial^\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\vec{\nabla} \end{pmatrix}, \quad A^\mu = \begin{pmatrix} \frac{\varphi}{c} \\ \vec{A} \end{pmatrix}$

b)

$$\vec{E}(\vec{r},t) = c\,g(y-ct)\,\vec{e}_x, \quad \vec{B}(\vec{r},t) = -g(y-ct)\,\vec{e}_z$$

i) $\vec{\nabla}\vec{B} = \vec{\nabla}\vec{E} = 0$

$$\vec{\nabla}\times\vec{B} = \vec{e}_x\partial_y B_z = -g'(y-ct)\vec{e}_x = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}$$

$$\vec{\nabla}\times\vec{E} = -\vec{e}_z\partial_y E_x = -c\,g'(y-ct)\vec{e}_z = -\frac{\partial\vec{B}}{\partial t}$$

ii)

$$\vec{S} = \frac{(\vec{E}\times\vec{B})}{\mu_o} = \frac{c}{\mu_o}g^2(y-ct)\,\vec{e}_y \qquad \left[\vec{e}_z\times\vec{e}_x = \vec{e}_y\right]$$

Aufgabe 4

a) $\varrho = \dfrac{q}{4\pi R^2}, \quad \vec{\jmath} = \varrho\,\vec{v}, \quad \vec{v} = \vec{\omega}\times\vec{r}$

$$\rightarrow \vec{\jmath} = \frac{q}{4\pi R^2}\,\delta(r-R)\;(\vec{\omega}\times\vec{r})$$

b)

$$\vec{m} = \frac{q}{8\pi\,R^2}\int \mathrm{d}^3r\,\delta(r-R)\;r\left[\vec{e}_r\times\big(\vec{\omega}\times(\rho\vec{e}_\rho + z\vec{e}_z)\big)\right]$$

$$\left[\vec{e}_z\times\vec{e}_\rho = \vec{e}_\phi, \quad \rho = r\sin\theta\right]$$

$$= \frac{q}{8\pi\,R^2}\int \mathrm{d}^3r\,\delta(r-R)\;r\left[\vec{e}_r\times(r\sin\theta\,\omega\vec{e}_\phi)\right] \qquad \left[\vec{e}_r\times\vec{e}_\phi = -\vec{e}_y\right]$$

$$= \frac{q\,\omega\,R^2}{8\pi}\int \mathrm{d}\Omega\sin\theta\,(-\vec{e}_y) \qquad \left[\vec{e}_y = \frac{1}{r}\frac{\partial\vec{r}}{\partial\theta}\right]$$

$$= \frac{q\,\omega\,R^2}{8\pi}\int\limits_{-1}^{1}\mathrm{d}\!\cos\theta\int\limits_{0}^{2\pi}\mathrm{d}\phi\sin\theta\,\begin{pmatrix}-\cos\theta\cos\phi\\ -\cos\theta\sin\phi\\ \sin\theta\end{pmatrix}$$

$$= \frac{q\,\omega\,R^2}{4}\,\vec{e}_z\underbrace{\int_{-1}^{1}\mathrm{d}\!\cos\theta\,(1-\cos^2\theta)}_{=\frac{4}{3}} = \frac{q\,R^2}{3}\;\vec{\omega}$$

c) Mit Hinweis (i) ist $\left|\vec{r}-\vec{r}'\right| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'}$

$$\vec{A}(\vec{r}) = \frac{q\,\mu_o}{16\pi^2 R^2}\left[\vec{\omega}\times\int \mathrm{d}^3r'\,\delta(r'-R)\frac{\vec{r}'}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}\right]$$

$$= \frac{q\,\mu_o\,R}{16\pi^2}\left[\vec{\omega}\times\int_{-1}^{1}\mathrm{d}\cos\theta'\int_0^{2\pi}\mathrm{d}\phi'\frac{1}{\sqrt{r^2+R^2-2rR\cos\theta'}}\begin{pmatrix}\sin\theta'\cos\phi'\\ \sin\theta'\sin\phi'\\ \cos\theta'\end{pmatrix}\right]$$

$$= \frac{q\,\mu_o\,R}{8\pi}\,(\vec{\omega}\times\vec{e}_{z'})\underbrace{\int_{-1}^{1}\mathrm{d}x\frac{x}{\sqrt{r^2+R^2-2rRx}}}_{=\begin{cases}\frac{2r}{3R^2}\,, & r<R\\ \frac{2R}{3r^2}\,, & r>R\end{cases}} \qquad \left[\vec{e}_{z'} = \vec{e}_r = \frac{\vec{r}}{r}\right]$$

$$\Rightarrow \vec{A}(\vec{r}) = \begin{cases}\frac{\mu_o\,q}{12\pi\,R}\,(\vec{\omega}\times\vec{r})\,, & r<R\\ \frac{\mu_o\,q\,R^2}{12\pi\,r^3}(\vec{\omega}\times\vec{r})\,, & r>R\end{cases}$$

d)

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times(\vec{\omega}\times\vec{r}) = \omega\underbrace{(\vec{\nabla}\vec{r})}_{=3} - \underbrace{(\vec{\omega}\cdot\vec{\nabla})\,\vec{r}}_{\omega_i\underbrace{\partial_i\vec{r}}_{\vec{e}_i}} = 3\vec{\omega}-\vec{\omega} = 2\vec{\omega}$$

$$\Rightarrow \vec{B}_{r<R} = \frac{\mu_o\,q}{6\pi\,R}\,\vec{\omega} = \frac{\mu_o}{2\pi}\frac{\vec{m}}{R^3}$$

e)

$$\vec{B}_{r<R} = \frac{\mu_o}{2\pi}\frac{\vec{m}}{R^3}\,, \qquad \vec{B}_{r>R} = \frac{\mu_o}{4\pi}\frac{(3\vec{e}_r\,(\vec{e}_r\vec{m})-\vec{m})}{r^3}$$

i) Normalkomponente: $(\vec{n} = \vec{e}_r)$

$$\vec{e}_r\cdot\left(\vec{B}_{r<R}-\vec{B}_{r>R}\right)\Big|_{r=R} = \frac{\mu_o}{2\pi\,R^2}(\vec{e}_r\vec{m}-\vec{e}_r\vec{m}) = 0$$

ii) Tangentialkomponente:

$$\vec{e}_r\times\left(\vec{B}_{r<R}-\vec{B}_{r>R}\right)\Big|_{r=R} = \frac{\mu_o}{4\pi\,R^3}\big(2(\vec{e}_r\times\vec{m})+(\vec{e}_r\times\vec{m})\big)$$

$$= \underbrace{\frac{3\mu_o}{4\pi\,R^3}\,(\vec{e}_r\times\vec{m})}_{\neq 0\ (\Rightarrow\ \text{unstetig})} = -\mu_o\underbrace{\frac{q}{4\pi\,R^2}\,(\vec{\omega}\times\vec{R})}_{\text{Flächenstrom }\vec{\kappa}}$$