1 Energie des elektrostatischen Feldes

Jackson Gl. (1.53): $W = \frac{1}{2}\int d^3x\, \rho(\vec{x})\,\Phi(\vec{x})$

$$= \frac{1}{8\pi\varepsilon_0}\int d^3x \int d^3x' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

(a) homogen geladene Kugel mit Radius R:

$$\rho(\vec{r}) = \theta(R-r)\,k\,, \quad \int d^3r\,\rho(r) = Q = \int_0^\infty dr \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, r^2 \sin\theta\, \rho(\vec{r})$$

$$\Leftrightarrow k = \frac{3Q}{4\pi}\frac{1}{R^3} \qquad = 4\pi\int_0^R dr\, r^2 k = \frac{4\pi}{3}R^3 k$$

$$W = \frac{1}{8\pi\varepsilon_0}\left(\frac{3Q}{4\pi}\frac{1}{R^3}\right)^2 (2\pi)^2 \int_0^R dr \int_0^R dr' \int_{-1}^{1} d\cos\theta \int_{-1}^{1} d\cos\theta' \frac{r^2 r'^2}{\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}}$$

alternativ (schneller): $W = \frac{\varepsilon_0}{2}\int d^3r\, |\vec{E}(\vec{r})|^2$

homogene Kugel: $\vec{E} = \frac{\vec{e}_r}{4\pi\varepsilon_0}\begin{cases} 1/r^2 & , r \geq R \\ r/R^3 & , r < R \end{cases}$ (Vorlesung?)

$$W = \frac{\varepsilon_0}{2}\frac{Q^2}{(4\pi\varepsilon_0)^2}\,4\pi\Big[\underbrace{\int_0^R dr\, \underbrace{r^2\left(\frac{r}{R^3}\right)^2}_{\frac{r^4}{R^6}}}_{\frac{1}{5R^6}r^5\big|_0^R} + \underbrace{\int_R^\infty dr\, \underbrace{\frac{r^2}{r^4}}_{\frac{1}{r^2}}}_{-\frac{1}{r}\big|_R^\infty}\Big]$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{R^5}{5R^6} - \underbrace{\frac{1}{\infty}}_{0} + \frac{1}{R}\right) = \frac{Q^2}{4\pi\varepsilon_0}\frac{3}{5}\frac{1}{R}$$

$$\frac{1}{5}+1 = \frac{6}{5}$$

(b) homogen geladene Kugelschale

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\,\vec{e}_r\,\frac{1}{r^2}\begin{cases} 0 & , r \le R_i \\ \dfrac{r^3 - R_i^3}{R_a^3 - R_i^3} & , R_i < r < R_a \\ 1 & , r \ge R_a \end{cases}$$

$$W = \frac{\varepsilon_0}{2}\int d^3r\,|\vec{E}(\vec{r})|^2 = \frac{\varepsilon_0}{2}\frac{Q^2}{(4\pi\varepsilon_0)^2}\cdot 4\pi\left[\int_0^{R_i} dr\, r^2\cdot 0 + \int_{R_i}^{R_a} dr\,\frac{r^2}{r^4}\left(\frac{r^3 - R_i^3}{R_a^3 - R_i^3}\right)^2 + \int_{R_a}^{\infty} dr\,\frac{r^2}{r^4}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{(R_a^3 - R_i^3)^2}\underbrace{\int_{R_i}^{R_a} dr\left(r^4 - 2r R_i^3 + \frac{R_i^6}{r^2}\right)}_{\frac{1}{5}r^5 - r^2 R_i^3 - \frac{R_i^6}{r}\Big|_{R_i}^{R_a}} + \underbrace{\int_{R_a}^{\infty} dr\,\frac{1}{r^2}}_{-\frac{1}{r}\Big|_{R_a}^{\infty}}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{\frac{R_a^5 - R_i^5}{5} - R_i^3(R_a^2 - R_i^2) + R_i^6\,\frac{R_a - R_i}{R_i R_a}}{(R_a^3 - R_i^3)^2} + \frac{1}{R_a}\right]$$

$\frac{1}{5} - 1 = -\frac{4}{5}$

$1 - \frac{4}{5} = \frac{1}{5}$

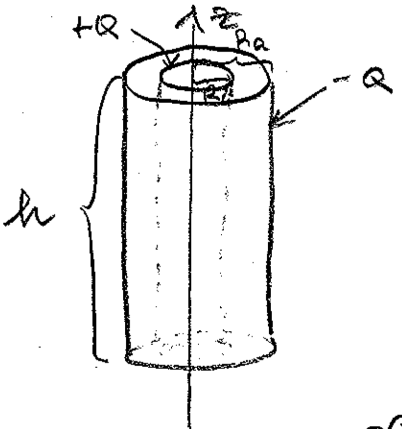
$\frac{1}{5} + 1 = \frac{6}{5}$

Grenzwerte: $R_i \to 0$: $W = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{\frac{1}{5}R_a^5 - \overset{0}{\cancel{R_a}} - 0}{R_a^6} + \frac{1}{R_a}\right]$

$= \frac{Q^2}{4\pi\varepsilon_0}\,\frac{3}{5}\,\frac{1}{R_a}$ wie homogen geladene Kugel

$R_i \to R_a$: $W = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{0 - 0 - 0}{0} + \frac{1}{R_a}\right] = \frac{Q^2}{4\pi\varepsilon_0}\,\frac{1}{2R_a}$

② Zylinderkondensator



Ladungsverteilung

$$\rho(\vec r) = k\left[k_1\delta(r-R_i) - k_2\delta(r-R_a)\right]\Theta\left(\frac{h}{2}-|z|\right)$$

$$\int_0^{R_a/2} dr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz\, r\, k\, \delta(r-R_i) = 2\pi k R_i h = Q$$

$$\rho(\vec r) = \frac{Q}{2\pi h}\left[\frac{\delta(r-R_i)}{R_i} - \frac{\delta(r-R_a)}{R_a}\right]\Theta\left(\frac{h}{2}-|z|\right)$$

$$\Phi(\vec r) = \int d^3r' \frac{\rho(\vec r')}{|\vec r-\vec r'|} = \frac{Q}{2\pi h}\cdot 2\pi \dots$$ (hässliches Integral)

Symmetrieüberlegungen: $\vec E(\vec r) = E_\rho \vec e_\rho$, 0 für $\rho > R_a$, 0 für $|z| > \frac{h}{2}$

Gaußscher Satz:

$$\frac{1}{\varepsilon_0}\underbrace{\int_V d^3r\, \rho(\vec r)}_{Q} = \underbrace{\int_{\partial V} \vec E \cdot d\vec f}_{2\pi h \rho E(\rho)}, \quad d\vec f = \rho\, d\varphi\, dz\, \vec e_\rho$$

$R_i \le \rho \le R_a$

$$\vec E(\vec r) = \begin{cases} 0, & \rho < R_i \\ \frac{Q}{2\pi\varepsilon_0 h}\frac{1}{\rho}, & R_i \le \rho < R_a \\ 0, & \rho \ge R_a \end{cases}, \quad \vec E = -\vec\nabla\Phi$$

$$\hookrightarrow E_\rho = -\frac{\partial\Phi}{\partial\rho}$$

$$\Phi(\rho) = -\int_{\rho_0}^{\rho} d\rho'\, E(\rho') + \Phi(\rho_0) = -\frac{Q}{2\pi\varepsilon_0 h}\underbrace{\int_{\rho_0}^{\rho}\frac{d\rho'}{\rho'}}_{\ln\rho - \ln\rho_0 = \ln\frac{\rho}{\rho_0}} + \Phi(\rho_0)$$

$$\boxed{\Phi(\rho) = \Phi(\rho_0) + \frac{Q}{2\pi\varepsilon_0 h}\ln\left(\frac{\rho_0}{\rho}\right)}$$ für $R_i \le \rho < R_a$, $\Phi(\rho < R_i)$, $\Phi(\rho > R_a)$

$$\Phi(\rho\to\infty) \overset{!}{=} 0 \Rightarrow \Phi(\rho > R_a) = 0 \Rightarrow \Phi(\rho = R_a) = 0 \overset{\rho_0 = R_a}{\Rightarrow} \Phi(R_i \le \rho < R_a) = \frac{Q}{2\pi\varepsilon_0 h}\ln\left(\frac{R_a}{\rho}\right)$$

const.

Φ stetig! $\qquad \Phi(\rho < R_i) = \frac{Q}{2\pi\varepsilon_0 h}\ln\left(\frac{R_a}{R_i}\right)$

(b) Kapazität: $C = \frac{Q}{U}$

anliegende Spannung: $U = \Phi(R_i) - \Phi(R_a)$

$$= \frac{Q}{2\pi\varepsilon_0 h} \ln\left(\frac{R_a}{R_i}\right) - 0$$

damit $$\boxed{C = \frac{Q}{U} = \frac{2\pi\varepsilon_0 h}{\ln(R_a/R_i)}}$$

(c) Gesamtenergie des Kondensators:

$$W = \frac{\varepsilon_0}{2} \int d^3x \, |\vec{E}|^2 = \frac{\varepsilon_0}{2} \left(\frac{Q}{2\pi\varepsilon_0 h}\right)^2 \underbrace{\int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz}_{2\pi h} \underbrace{\int_{R_i}^{R_a} d\rho \, \rho \frac{1}{\rho^2}}_{\ln\rho\big|_{R_i}^{R_a}}$$

$$\boxed{= \frac{Q^2}{4\pi\varepsilon_0 h} \ln(R_a/R_i)}$$

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③ Multipolentwicklung

(I)

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \dots\right]$$

kartesisch: $Q = \int d^3r'\, \rho(\vec{r}\,')$ Monopolmoment

$p_i = \int d^3r'\, \rho(\vec{r}\,')\, x_i$ Dipolmoment

$Q_{ij} = \int d^3r'\, \rho(\vec{r}\,')\left(3x_i' x_j' - r'^2\delta_{ij}\right)$ Quadrupolmoment

(a) (i) $\rho(\vec{r}) = q\,\delta(x)\,\delta(y)\,\delta(z-d) - q\,\delta(x)\,\delta(y)\,\delta(z)$

$Q = \int d^3r'\, \rho(\vec{r}\,') = q - q = 0$

$p_i = \int d^3r'\, \rho(\vec{r}\,')\, x_i = dq - 0\cdot q = dq$ für $i=3$, $i=1,2$: $p_{1,2}=0$

$$Q_{ij} = \int d^3r'\, \rho(\vec{r}\,')\big(3x_i'x_j' - \underbrace{r'^2}_{x'^2+y'^2+z'^2}\delta_{ij}\big) = 3\underbrace{\int d^3r'\, \rho(\vec{r}\,')\, x_i' x_j'}_{i,j\in\{1,2\}\to 0} - \underbrace{\int d^3r'\, \rho(\vec{r}\,')\, r'^2\delta_{ij}}_{= q d^2\delta_{ij}\ \forall i,j}$$

$\Rightarrow Q_{33} = 3qd^2 - qd^2 = 2qd^2$

$Q_{11} = Q_{22} = -qd^2$, $Q_{ij} = 0$ für $i \neq j$

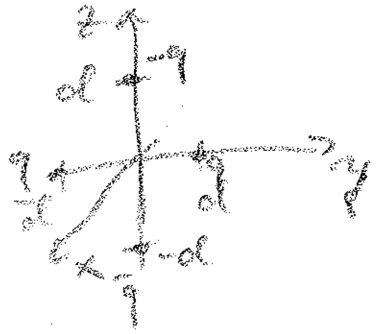
Nebenbemerkung: $\mathrm{Sp}\, Q_{ij} = Q_{11} + Q_{22} + Q_{33} = 0$ (gilt immer!)

(b) (i) Potential (s.o.): $\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{0}{r} + \frac{dqz}{r^3} + \frac{qd^2}{2}\left(\frac{\overbrace{-x^2-y^2}^{-\rho^2}+2z^2}{r^5}\right)\right]$

Zylinderkoord.: $\vec{\nabla} = \vec{e}_\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho}\vec{e}_\varphi\frac{\partial}{\partial\varphi} + \vec{e}_z\frac{\partial}{\partial z}$, $|\vec{r}| = \sqrt{\rho^2+z^2}$

$$\vec{E} = -\vec{\nabla}\Phi = \frac{qd}{4\pi\varepsilon_0}\left[\left(\frac{3}{r^5}\left(z-\frac{d}{2}\right) + \frac{5}{r^7}\,\frac{3}{2}z^2 d\right)\vec{r} - \frac{1}{r^3}\left(\frac{3dz}{r^2}+1\right)\vec{e}_z\right]$$

(a) (ii) $\rho(\vec{r}\,') = q\,\delta(x)\big[\delta(y-d)\,\delta(z) + \delta(y+d)\,\delta(z)$
$\qquad - \delta(y)\,\delta(z-d) - \delta(y)\,\delta(z+d)\big]$



$$Q = \int d^3r'\,\rho(\vec{r}\,') = q(1+1-1-1) = 0$$

$$p_i = \int d^3r'\,\rho(\vec{r}\,')\,x_i = \begin{cases} 0 & i=1 \\ q(d-d)=0 & i=2 \\ q(-d-(-d))=0 & i=3 \end{cases} \Bigg\} \equiv 0$$

$$Q_{ij} = \int d^3r'\,\rho(\vec{r}\,')\,(3x_i' x_j' - r'^2\,\delta_{ij})$$

$$Q_{xx} = -q d^2 \cdot 0 \;, \quad Q_{yy} = q\,(3d^2 \cdot 2 - 0\,d^2) = 6qd^2$$

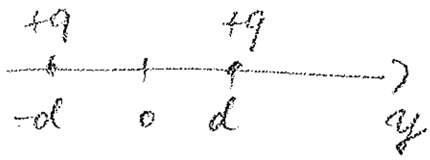
$$Q_{zz} = -q\,(3d^2 \cdot 2 - 0\,d^2) = -6qd^2$$

$$Q_{xi} \underset{i \neq x}{=} 0 \;, \quad Q_{yz} = -q \cdot 0 = 0$$

(b) (ii) $\Phi(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\,\dfrac{1}{2}\cdot 6qd^2\left(\dfrac{y^2-z^2}{r^5}\right) = \dfrac{3qd^2}{4\pi\varepsilon_0}\,\dfrac{y^2-z^2}{r^5}$

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi = \frac{3qd^2}{4\pi\varepsilon_0}\Bigg[\left(\frac{2y}{r^5} - \frac{5}{2}\,\frac{2y(y^2-z^2)}{r^7}\right)\vec{e_y} - \left(\frac{2z}{r^5} - \frac{5}{2}\,\frac{2z(y^2-z^2)}{r^7}\right)\vec{e_z}$$
$$- 5\,\frac{x(y^2-z^2)}{r^7}\,\vec{e_x}$$

$$= \frac{3qd^2}{4\pi\varepsilon_0}\left[\frac{5(y^2-z^2)}{r^7}\,\vec{r} - \frac{2y}{r^5}\,\vec{e_y} + \frac{2z}{r^5}\,\vec{e_z}\right]$$



$$Q = q\int_{-\infty}^{\infty} dy\, \underbrace{\delta(|y|-d)}_{\delta(\sqrt{y^2}-d)} \overset{?}{=} 2q$$

Nullstellen: $\sqrt{y^2} - d = 0 \underset{d>0}{\Leftrightarrow} y^2 = d^2$

$\Rightarrow y_{1,2} = \pm d$

$$\frac{d}{dy}\left(\sqrt{y^2} - d\right) = \frac{1}{2}\frac{2y}{\sqrt{y^2}}$$

$$= \frac{y}{\sqrt{y^2}} = \mathrm{sign}(y)$$ (gibt nur das Vorzeichen des Arguments)

$$\delta(f(x)) = \sum \frac{1}{|f'(x_0)|}\delta(x-x_0)$$

$$\Rightarrow \delta(\sqrt{y^2} - d) = \frac{1}{\mathrm{sign}(d)}\delta(y+d) + \frac{1}{\mathrm{sign}(d)}\delta(y-d)$$

$$\Rightarrow \int_{-\infty}^{\infty} dy\, \delta(|y|-d) = \int_{-\infty}^{\infty} dy\, \delta(y+d) + \int_{-\infty}^{\infty} dy\, \delta(y-d)$$

✓

Best regards,
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$\Phi(\vec{r}) = \frac{qd}{4\pi\varepsilon_0}\left[\frac{z}{r^3} + \frac{d}{2}\left(\frac{2z^2-\varrho^2}{r^5}\right)\right]$

$\frac{\partial\Phi}{\partial z} = \frac{qd}{4\pi\varepsilon_0}\left[\frac{1}{r^3} - 3\frac{z^2}{r^5} + \frac{2dz}{r^5} - 5\frac{dz}{2r^7}(2z^2-\varrho^2)\right]$

$\frac{\partial\Phi}{\partial\varrho} = \frac{qd}{4\pi\varepsilon_0}\left[-3\frac{z\varrho}{r^5} - \frac{d\varrho}{r^5} - 5\frac{d\varrho}{2r^7}(2z^2-\varrho^2)\right]$ $\qquad -\frac{d\varrho}{r^5} + \frac{2dz}{r^5} = -\frac{\cancel{d\varrho}}{r^5} + \frac{3dz}{r^5}$

$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi = +\frac{qd}{4\pi\varepsilon_0}\left[\left(\frac{3z}{r^5} + \frac{5d}{2}\underbrace{\frac{2z^2-\varrho^2}{r^7}} + \frac{d}{r^5}\right)\vec{r} - \left(\frac{3dz}{r^5} + \frac{1}{r^3}\right)\vec{e_z}\right]$

$\frac{3z^2-(z^2+\varrho^2)}{r^7} = \frac{3z^2}{r^7} - \frac{1}{r^5}$

$-\frac{3}{2}+1 = -\frac{1}{2}$

$= \frac{qd}{4\pi\varepsilon_0}\left[\left(\frac{3}{r^5}\left(z-\frac{d}{2}\right) + \frac{15dz^2}{2r^7}\right)\vec{r} - \left(\frac{3dz}{r^5} + \frac{1}{r^3}\right)\vec{e_z}\right]$

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