Karlsruher Institut für Technologie

Theoretische Physik C — WS 2013/14 ①

4. Übungsblatt

① Green'sche Funktion des harmonischen Oszillators

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)G(t,t')=\delta(t-t') \quad , \quad G(t,t')=0 \; (t<t')$$

(a) Green'sche Funktion: Spezielle Lsg. der DGL mit δ-Funktion als Inhomogenität.

(i) Ansatz: $G(t,t')=G(t-t')$

$$\underbrace{\int_{t'-\varepsilon}^{t'+\varepsilon} dt\left[\left(\frac{d^2}{dt^2}+\omega_0^2\right)G(t,t')\right]} = \int_{t'-\varepsilon}^{t'+\varepsilon} dt\,\delta(t-t')=1$$

$$\underbrace{\dot{G}(t-t')\Big|_{t=t'-\varepsilon}^{t=t'+\varepsilon}} + \underbrace{\int_{t'-\varepsilon}^{t'+\varepsilon} dt\, G(t-t')}$$

$$\underbrace{\dot{G}(t'+\varepsilon-t')}_{\dot{G}(\varepsilon)} - \underbrace{\dot{G}(t'-\varepsilon+t')}_{=0 \;!!} \qquad \rightarrow 0 \; (\lim \varepsilon\to 0)$$

$$\Rightarrow \boxed{\lim_{\varepsilon\to 0}\dot{G}(\varepsilon)=1}$$

(ii) Anfangsbedingungen: $\dot{G}(0)=1$ (wie in (i) gezeigt)
$G(t-t'\leq 0)=0 \rightarrow G\sim\Theta(t-t')$

$t-t'>0$: $\ddot{G}(t-t')+\omega_0^2 G(t-t')=0$

homogene Gl. $\rightarrow$ Lösung $G(\tilde{t})=\left[A\sin(\omega_0\tilde{t})+B\cos(\omega_0\tilde{t})\right]\Theta(\tilde{t})$

$G(0)=0 \rightarrow B=0$, $\dot{G}(0)=1 \rightarrow A\omega_0=1 \rightarrow A=\frac{1}{\omega_0}$

$$G(\tilde{t})=\frac{\Theta(\tilde{t})}{\omega_0}\sin(\omega_0\tilde{t})=\frac{1}{\omega_0}\frac{1}{2i}\Theta(\tilde{t})\left[e^{i\omega_0\tilde{t}}-e^{-i\omega_0\tilde{t}}\right]$$

$\tilde{t}=t-t'$

(b) allgemeine Lösung für getriebenen harmon. Osz.:

$$m\left(\frac{d^2}{dt^2} + \omega_0^2\right) x(t) = F(t)$$

für $F(t) = F_0\, t\, \Theta(t)\, \Theta(t_0 - t)$, $t_0 > 0$

und Anfangsbed. $x(0) = 0$, $\dot{x}(0) = 0$.

$$\ddot{x}(t) + \omega_0^2 x(t) = \frac{1}{m} F(t) \equiv f(t)$$

$$x(t) = \int_{-\infty}^{\infty} dt'\, f(t')\, G_r(t-t') + \underbrace{x_0(t)}_{\text{Integrationskonstante in } x \text{ (homogene Lösung)}}$$

$t < t_0$: $$x(t) = \frac{1}{\omega_0} \int_{-\infty}^{\infty} dt' \sin(\omega_0(t-t'))\, \Theta(t-t')\, f_0 t'\, \Theta(t')\, \Theta(t_0 - t')$$

$$= \frac{f_0}{\omega_0} \int_0^t dt'\, t' \sin(\omega_0(t-t')) = \frac{f_0}{\omega_0} \frac{\omega_0 t - \sin(\omega_0 t)}{\omega_0^2}$$

Anf.bed.: $\lim\limits_{t\to 0} x(t) = 0$ ✓ $\lim\limits_{t\to 0} \dot{x}(t) = \frac{f_0}{\omega_0}\left[\frac{1}{\omega_0} - \cos(\omega_0 t)\right] = 0$

$t > t_0$: $$x(t) = \frac{f_0}{\omega_0} \int_{-\infty}^{\infty} dt' \sin(\omega_0(t-t'))\, \Theta(t-t')\, f_0 t'\, \Theta(t')\, \Theta(t_0 - t')$$

$$= \frac{f_0}{\omega_0} \int_0^{t_0} dt'\, t' \sin(\omega_0(t-t')) = f_0 \frac{t_0 \cos(\omega_0(t-t_0))}{\omega_0}$$

$$+ \frac{1}{\omega_0^2}\left(\sin(\omega_0(t-t_0)) - \sin(\omega_0 t)\right)$$

$$\lim_{t\to t_0} x_{(ii)}(t) = \lim_{t\to t_0} x_{(i)}(t)$$

✓

$$\lim_{t\to t_0} \dot{x}_{(ii)}(t) = \lim_{t\to t_0} \dot{x}_{(i)}(t)$$

③

② Greensche Funktion in zwei Dimensionen

$$\boxed{\Delta_{\vec{r}}\, G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')}$$

Ansatz: $G(\vec{r},\vec{r}\,') = G(|\vec{r}-\vec{r}\,'|)$

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

Lösung für $\rho \neq 0$: $\Delta G(\rho) = 0 = \underbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) G(\rho)}_{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho G'(\rho))} = \frac{1}{\rho}G'(\rho) + G''(\rho) = 0$

$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho G'(\rho)) \Leftrightarrow \rho G'(\rho) = \text{const.}$

$$\int_0^G dG' = \int_{\rho_0}^{\rho}\frac{d\rho'}{\rho'} \rightarrow G(\rho) = c\ln\left(\rho/\rho_0\right)$$

Gaußscher Satz: $\int_V \underbrace{\vec{\nabla}\cdot(\vec{\nabla}G)}_{\Delta G = \delta(\vec{r}-\vec{r}\,')} dV = \underbrace{\int_{\partial V} \vec{\nabla}G \cdot d\vec{A}}$

$\int dx \int dy \int dz\, \delta(x-x')\delta(y-y') = \delta(\rho\vec{e}_\rho)$ $\qquad \int dz \int_0^{2\pi} d\varphi\, \rho \cdot c\,\frac{\rho_0}{\rho}\frac{1}{\rho_0}\vec{e}_\rho^{\,2} \overset{c}{=} 2\pi \int dz$

$= \int dz$

$\Leftrightarrow \int dz = 2\pi c \int dz \quad \Leftrightarrow \boxed{c = \frac{1}{2\pi}}$

unbestimmtes Integral: $\int_{-z}^{z} dz'$ über festes Intervall entlang z-Achse, wobei $z \rightarrow \varepsilon$ klein ...

$$\boxed{G(\rho) = \frac{1}{2\pi}\ln\left(\rho/\rho_0\right)}$$

③ Die Poisson-Gleichung der Elektrostatik

$$\Delta \Phi(\vec{r}) = -\frac{1}{\varepsilon_0}\rho(\vec{r})$$

(a) Greensche Funktion $G(\vec{r}-\vec{r}')$ der Poisson-Gl.

$$\Delta_{\vec{r}}\, G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$$

(i) Fouriertransformierte $\mathcal{F}[G] = \hat{G}(\vec{k})$ mit

$$\mathcal{F}[f] = \hat{f}(\vec{k}) = \frac{1}{\sqrt{(2\pi)^3}}\int d^3r\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

und $\mathcal{F}^{-1}[\hat{f}] = f(\vec{r}) = \frac{1}{\sqrt{(2\pi)^3}}\int d^3k\, \hat{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$

- F.T. der δ-Funktion

$$\mathcal{F}[\delta] = \hat{\delta}(\vec{k}) = \frac{1}{\sqrt{(2\pi)^3}}\int d^3r\, \delta(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} = \frac{1}{\sqrt{(2\pi)^3}}\, e^0$$

- Fourierdarstellung der δ-Funktion:

$$\delta(\vec{r}-\vec{r}') = \frac{1}{\sqrt{(2\pi)^3}}\int d^3k\, \underbrace{\hat{\delta}(\vec{k})}_{\frac{1}{\sqrt{(2\pi)^3}}}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \quad (*)$$

- $$\mathcal{F}^{-1}[\mathcal{F}[f]] = \frac{1}{(2\pi)^3}\int d^3k \int d^3r'\, f(\vec{r}')\, \underbrace{e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}'}}_{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}$$

$$\overset{(*)}{=} \int d^3r'\, f(\vec{r}')\, \delta(\vec{r}-\vec{r}') = f(\vec{r})$$

$$\mathcal{F}[G] \rightarrow \quad \Delta_{\vec{r}} \underbrace{\int d^3k\, \hat{G}(\vec{k})\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}_{\mathcal{F}^{-1}[\hat{G}]} = \mathcal{F}^{-1}[\delta] = \int d^3k\, \frac{1}{\sqrt{(2\pi)^3}}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$\int d^3k \left[-k^2\hat{G}(\vec{k}) - \frac{1}{\sqrt{(2\pi)^3}}\right] e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = 0 \quad \text{gilt } \forall\, \vec{r}, \vec{r}'$$

①

$$\Rightarrow \boxed{\hat{G}(\vec{k}) = -\frac{1}{\sqrt{(2\pi)^3}\,\vec{k}^2}}$$

(ii) $$G(\vec{r}-\vec{r}\,') = -\frac{1}{(2\pi)^3}\underbrace{\int d^3k}_{\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_0^\infty k^2\,dk} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{\vec{k}^2}$$

$\int_0^{2\pi} d\varphi \to 2\pi$

$\vec{r}-\vec{r}\,' = |\vec{r}-\vec{r}\,'|\,\vec{e}_z = \eta\,\vec{e}_z$

$\vec{k}(\vec{r}-\vec{r}\,') = \eta k \cos\theta$

$$= -\frac{1}{(2\pi)^2}\int_0^\infty dk \int_{-1}^{1} d\cos\theta\, \underbrace{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}_{\exp(ik\eta\cos\theta)}$$

$$= -\frac{1}{(2\pi)^2}\int_0^\infty dk \underbrace{\left[\frac{e^{ik\eta}}{ik\eta} - \frac{e^{-ik\eta}}{ik\eta}\right]}_{2\,\frac{\sin(k\eta)}{k\eta}} = -\frac{1}{(2\pi)^2}\cdot 2\cdot\frac{\pi}{2\eta} = -\frac{1}{4\pi\eta}$$

$$\int_0^\infty dx\,\frac{\sin(ax)}{x} = \frac{\pi}{2}$$

$$= \frac{-1}{4\pi}\,\frac{1}{|\vec{r}-\vec{r}\,'|}$$

(b) allgemeine Lösung der Poisson-Gl.: $\Delta\Phi(\vec{r}) = -\frac{1}{\varepsilon_0}\rho(\vec{r})$

$$\rho(\vec{r}) = \int d^3r'\,\delta(\vec{r}-\vec{r}\,')\,\rho(\vec{r}\,')$$

$$\Rightarrow \Phi(\vec{r}) = -\frac{1}{\varepsilon_0}\int d^3r'\,G(\vec{r}-\vec{r}\,')\,\rho(\vec{r}\,')$$

$$\left(\text{da } \Delta\Phi(\vec{r}) = -\frac{1}{\varepsilon_0}\int d^3r'\,\underbrace{\Delta_{\vec{r}}\,G(\vec{r}-\vec{r}\,')}_{\delta(\vec{r}-\vec{r}\,')}\,\rho(\vec{r}\,')\right)$$

$$\boxed{\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

(spezielle Lösung für inhomogenität $\rho(\vec{r})$)

allgemeine Lösung: + homogene Lösung

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + F(\vec{r}),\ \text{wobei } \Delta F(\vec{r}) = 0$$