Theoretische Physik C – WS 2013/14 ①
Elektrodynamik
7. Übungsblatt

① Sphärische Multipolentwicklung

$$\rho(\vec{r}) = \frac{q}{64\pi a_0^5}\, r^2 e^{-r/a_0} \sin^2\theta \,, \quad a_0 > 0$$

(a) Entwicklung nach sphärischen Multipolen

$$q_{lm} = \int d^3r'\, \rho(\vec{r}\,')\, r'^l\, Y^*_{lm}(\theta', \varphi') \qquad (*)$$

drücke winkelabhängigen Teil durch Kugelflächenfunktionen aus: (suche ein Y_{lm}, das einen $\sin^2\theta$-Beitrag enthält – kein $e^{i\varphi}$, kein $\cos\theta$ ⇝ Jackson 3.5)

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\underbrace{\cos^2\theta}_{1-\sin^2\theta} - \frac{1}{2}\Big) = \sqrt{\frac{5}{4\pi}}\Big(1 - \frac{3}{2}\sin^2\theta\Big)$$

$$\Leftrightarrow \sin^2\theta = -\sqrt{\frac{4\pi}{5}}\,\frac{2}{3}\, Y_{20}(\theta,\varphi) + \underbrace{\frac{2}{3}}_{\frac{2\sqrt{4\pi}}{3} Y_{00}(\theta,\varphi)} \qquad \Big\}\ Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow q_{lm} = \frac{q}{64\pi a_0^5} \int_0^\infty dr'\, \underbrace{\underbrace{r'^l}_{(*)}\, \underbrace{r'^2}_{\rho(\vec r\,')}\, \underbrace{r'^2}_{\text{Jacobi}}}_{(r')^{l+4}}\, e^{-r/a_0} \underbrace{\int d\Omega'\, Y^*_{lm}(\theta',\varphi')\, \Big[Y_{00}(\theta',\varphi') - \frac{1}{\sqrt{5}} Y_{20}(\theta',\varphi')\Big]\frac{2}{3}\sqrt{4\pi}}_{\frac{2}{3}\sqrt{4\pi}\,(\delta_{l0}\delta_{m0} - \frac{1}{\sqrt{5}}\delta_{l2}\delta_{m0})}$$

Orthogonalität

$$\int_0^\infty dr'\,(r')^{l+4} e^{-r/a_0} \underset{\substack{(l+4)\text{ mal} \\ \text{partiell integrieren}}}{=} (l+4)!\, \underbrace{(+1)^{l+4}}_{+1}\, a_0^{l+4} \int_0^\infty dr'\, e^{-a/r_0}$$

$$= (l+4)!\, a_0^{l+5}$$

$$\Rightarrow q_{00} = \frac{q}{64\pi a_0^5}\,\frac{2}{3}\sqrt{4\pi}\,(0+4)!\, a_0^{0+5} = \frac{q}{2\sqrt{\pi}}$$

$$q_{20} = \frac{q}{64\pi a_0^5}\,\frac{2}{3}\sqrt{4\pi}\,(2+4)!\, a_0^{2+5}\Big(\frac{-1}{\sqrt{5}}\Big) = \frac{-15}{\sqrt{5\pi}}\, q a_0^2$$

(b) Potential für große r nach Legendre-Polynomen entwickeln

azimutale Symmetrie
↳ keine φ-Abhängigkeit

(2)

$$\Phi(\vec r) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec r')}{|\vec r - \vec r'|} = \frac{1}{\varepsilon_0}\sum_{l,m}\frac{1}{2l+1} Y_{lm}(\theta,\varphi)\int d^3r' \rho(\vec r')\frac{r_<'^{\,l}}{r_>'^{\,l+1}} Y^*_{lm}(\theta',\varphi')$$

$r_<' = \min(r,r')$, $r_>' = \max(r,r')$

Entwicklung für große r ↳ nur $\frac{r'^l}{r^{l+1}}$ relevant

da $\int_0^\infty dr' r'^2 \rho(\vec r')\frac{r_<'^{\,l}}{r_>'^{\,l+1}} = \int_0^r dr' \frac{r'^{2+l}}{r^{l+1}}\rho(\vec r') + \underbrace{\int_r^\infty dr' r'^2 \frac{r^l}{r'^{l+1}}\rho(\vec r')}_{\underbrace{r^l\int_r^\infty dr'(r')^{1-l} r'^2 e^{-r'/a_0}}_{\sim e^{-r/a_0} \text{ f. } r\to\infty}}$

$r\to\infty$

$$\Phi(\vec r) = \frac{1}{\varepsilon_0}\sum_{l,m}\frac{Y_{lm}(\theta,\varphi)}{2l+1}\frac{1}{r^{l+1}}\underbrace{\int_0^\infty dr' \frac{r'^2 r'^l}{(r')^{2+l}}\int d\Omega' Y^*_{lm}(\theta',\varphi')\rho(\vec r')}_{q_{lm} = \delta_{l0}\delta_{m0}\underbrace{\frac{q}{2\sqrt\pi}}_{q_{00}} - \delta_{l2}\delta_{m0}\underbrace{\frac{15}{\sqrt{5\pi}}}_{-q_{20}} q a_0^2}$$

$$\boxed{\Phi(\vec r) = \frac{1}{\varepsilon_0} Y_{00}(\theta,\varphi)\frac{q_{00}}{r} + \frac{1}{\varepsilon_0} Y_{20}(\theta,\varphi)\frac{q_{20}}{5 r^3}}$$

kartesische Multipole: $\Phi(\vec r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{|\vec r|} + \frac{\vec p\cdot\vec r}{|\vec r|^3} + \frac{1}{2}\sum_{ij}\frac{x_i Q_{ij} x_j}{|\vec r|^5} + \dots\right]$

ablesen: $\frac{1}{4\pi}\frac{Q}{r} = Y_{00}\frac{q_{00}}{r} = \frac{1}{\sqrt{4\pi}}\frac{q}{2\sqrt\pi\, r} \Rightarrow \boxed{Q = q}$

$\vec p = 0$ (kein Term $\sim r^{-2}$)

$$\frac{1}{r^2}\underbrace{\sum_{ij} x_i Q_{ij} x_j}_{x^2 Q_{xx} + y^2 Q_{yy} + z^2 Q_{zz}} = 2\cdot 4\pi\, Y_{20}\frac{-15}{\sqrt{5\pi}}\frac{q a_0^2}{5} = 2\cdot 4\pi\sqrt{\frac{5}{4\pi}}\frac{-15}{\sqrt{5\pi}}\frac{q a_0^2}{5}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$= -6 q a_0^2 (3\cos^2\theta - 1)$

$= -6\frac{q a_0^2}{r^2}(3z^2 - r^2)$

azimutale Symm.: $Q_{xx} = Q_{yy}$

$Q_{zz} = -2Q_{xx}$ ↳ spurlos

$\frac{1}{2}Q_{zz}(2z^2 - x^2 - y^2) = Q_{zz}(3z^2 - r^2)$ ⟶ $Q_{zz} = -\cancel{12}\, q a_0^2 = -2 Q_{xx}$

$= -2 Q_{yy}$

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Entwicklung nach Kugelflächenfunktionen:

$$\Phi(\vec r) = \frac{1}{\varepsilon_0}\sum_{lm} \frac{Y_{lm}(\theta,\varphi)}{2l+1}\int d^3r'\,\rho(\vec r')\,\frac{r_<'^{\,l}}{r_>'^{\,l+1}}\, Y^*_{lm}(\theta',\varphi')$$

$r_<' = \min(r,r')$, $r_>' = \max(r,r')$

(c) Nahe $r=0$

$$\Phi(\vec r) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{Y_{lm}(\theta,\varphi)}{2l+1}\, r^l \int_r^\infty dr'\,(r')^{1-l}\int d\Omega'\, Y^*_{lm}(\theta',\varphi')\,\rho(\vec r')$$

$$= \frac{1}{\varepsilon_0}\sum_{lm}\frac{Y_{lm}(\theta,\varphi)}{2l+1}\, r^l\int_r^\infty dr'\,(r')^{3-l}\, e^{-r'/a_0}\left(\delta_{l0}\delta_{m0}-\frac{1}{\sqrt5}\delta_{l2}\delta_{m0}\right)\frac{2\sqrt{4\pi}\,q}{3\cdot 64\pi a_0^5}$$

$$= \frac{1}{\varepsilon_0}\frac{q}{64\pi a_0^5}\frac{2}{3}\sqrt{4\pi}\left\{ Y_{00}\int_r^\infty dr'\, r'^3 e^{-r'/a_0} - \frac{1}{5}\frac{1}{\sqrt5}Y_{20}\, r^2 \underbrace{\int_r^\infty dr'\, r' e^{-r'/a_0}}_{\substack{-a_0 r' e^{-r'/a_0}\big|_r^\infty + a_0\int_r^\infty dr' e^{-r'/a_0} \\ = a_0 r e^{-r/a_0} + a_0^2 e^{-r/a_0}}} \right\}$$

$$= \frac{q}{48\sqrt{\pi}\,\varepsilon_0 a_0^5}\left[Y_{00}\, a_0\left(6a_0^3+6a_0^2 r+3a_0 r^2 + r^3\right)e^{-r/a_0} - \frac{r^2}{5\sqrt5}Y_{20}\, a_0(r+a_0)\,e^{-r/a_0}\right]$$

entwickle $e^{-r/a_0} = 1 - \frac{r}{a_0} + \frac{r^2}{2a_0^2} - \frac{r^3}{6a_0^3} + \dots$

damit: $a_0\left(6a_0^3+6a_0^2r+3a_0r^2+r^3\right)\left(1-\frac{r}{a_0}+\frac{r^2}{2a_0^2}-\frac{r^3}{6a_0^3}+\dots\right)$
$= 6a_0^4 + \mathcal{O}(r^4)$

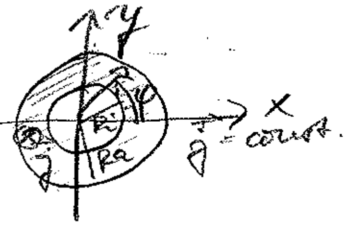
ebenso $r^2 a_0(a_0+r)\,e^{-r/a_0} = a_0^2 r^2 + \mathcal{O}(r^4)$

$$= \frac{q}{48\sqrt{\pi}\,\varepsilon_0 a_0^5}\left[\underbrace{Y_{00}}_{\frac{1}{\sqrt{4\pi}}}\cdot 6a_0^4 - \frac{1}{5\sqrt5}Y_{20}\, r^2 a_0^2 + \mathcal{O}(r^4)\right]$$

$$= \frac{q}{96\pi\varepsilon_0}\left[\frac{6}{a_0} - \frac{1}{5}\frac{r^2}{a_0^3}\underbrace{\left(\frac32\cos^2\theta-\frac12\right)}_{P_2(\cos\theta)} + \mathcal{O}(r^4)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 a_0}\left[\frac14 - \frac{1}{120}\frac{r^2}{a_0^2}P_2(\cos\theta) + \mathcal{O}(r^4)\right]$$

② Zylinderförmiger Hohlleiter



(a) magnetische Induktion mittels

Biot-Savart: $\vec B(\vec r) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec j(\vec r')\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}$

Rotationssymmetrie um z-Achse: $\vec B$ φ-unabhängig

↳ $\vec r = \rho\vec e_\rho + z\vec e_z$, $\vec j = j_0 \vec e_z$ mit j_0 = const.

$|\vec r-\vec r'| = \left(\rho^2+\rho'^2 - 2\rho\rho'\cos\varphi' + (z-z')^2\right)^{1/2}$

↑ wähle o.B.d.A. $\varphi = 0$

$\vec j\times\vec r = j_0 \vec e_z\times(\rho\vec e_\rho + z\vec e_z) = j_0\rho\vec e_\varphi$

$$\vec B(\vec r) = \frac{\mu_0 j_0}{4\pi}\int_{R_i}^{R_a} d\rho'\,\rho'\int_0^{2\pi}d\varphi'\int_{-\infty}^{\infty}dz' \, \frac{\rho\vec e_\varphi - \rho'\vec e_{\varphi'}}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\varphi' + (z-z')^2}^{\,3}}$$

$$= \frac{\mu_0 j_0}{4\pi}\int_{R_i}^{R_a} d\rho'\,\rho'\underbrace{\int_0^{2\pi}d\varphi' \, \frac{2(\rho\vec e_\varphi-\rho'\vec e_{\varphi'})}{\rho^2+\rho'^2-2\rho\rho'\cos\varphi'}}$$

, $\vec e_{\varphi'} = \begin{pmatrix}-\sin\varphi'\\ \cos\varphi'\\ 0\end{pmatrix}$

$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}\Big|_{-\infty}^{\infty} = \frac{2}{a^2}$

$\vec e_\varphi = \vec e_y$ wg. $\varphi = 0$

$\rho\vec e_\varphi - \rho'\vec e_{\varphi'} = \rho'\sin\varphi'\,\vec e_{x'} + (\rho-\rho'\cos\varphi')\,\vec e_{y'}$

$$\vec e_{y'}\int_0^{2\pi}d\varphi' \, \frac{\rho-\rho'\cos\varphi'}{\rho^2+\rho'^2-2\rho\rho'\cos\varphi'} = \vec e_{y'}\int_0^{2\pi}d\varphi' \, \frac{\frac{1}{\rho}(\rho^2+\rho'^2-2\rho\rho'\cos\varphi') + (\rho^2-\rho'^2)\frac{1}{\rho}}{\rho^2+\rho'^2-2\rho\rho'\cos\varphi'}$$

$= \frac{2\pi}{\rho} + \frac{\rho^2-\rho'^2}{\rho}\,\frac{2\pi}{|\rho^2-\rho'^2|}$ da $\rho^2+\rho'^2-2\rho\rho'\cos\varphi = |\rho^2-\rho'^2|$

$$\vec B(\vec r) = \frac{\mu_0 j_0}{4\pi}\vec e_{y'}\int_{R_i}^{R_a}d\rho'\,\rho'\left[\frac{2\pi}{\rho} + \frac{\rho^2-\rho'^2}{\rho}\,\frac{2\pi}{|\rho^2-\rho'^2|}\right]$$

$= \frac{\mu_0 j_0}{4\pi}\vec e_{y'}\int_{R_i}^{R_a}d\rho'\,\rho'\,\frac{4\pi}{\rho}\,\Theta(\rho-\rho')$ da $\frac{x}{|x|} = \mathrm{sgn}(x)$

↳ $\frac{1}{\rho} - \frac{1}{\rho}\,\frac{\rho^2-\rho'^2}{|\rho^2-\rho'^2|} = \frac{2}{\rho}\Theta(\rho-\rho')$

$$\vec B(\vec r) = \begin{cases} 0 & \text{für } \rho < R_i \\ \vec e_{y'}\,\frac{\mu_0 j_0}{4\pi}\,\frac{4\pi}{\rho}\,\frac{R_a^2-R_i^2}{2} = \frac{\mu_0 j_0}{2}(R_a^2-R_i^2)\vec e_{y'} & \text{für } \rho > R_a \\ \vec e_{y'}\,\frac{\mu_0 j_0}{4\pi}\,\frac{4\pi}{\rho}\,\frac{\rho^2-R_i^2}{2} = \frac{\mu_0 j_0}{2}(\rho^2-R_i^2)\,\vec e_{y'} & \text{für } R_i < \rho < R_a \text{ (obere Grenze} = \rho) \end{cases}$$

(b) Symmetrieüberlegungen: $\vec{B}(\vec{r}) = B_\rho \vec{e}_\rho + B_\varphi \vec{e}_\varphi + B_z \vec{e}_z$

wobei $\vec{B}(\vec{r}) = \vec{B}(\rho)$ (keine Abhängigkeit von φ und z)

Maxwell-Gleichungen:

$\vec{\nabla}\cdot\vec{B} = 0 \to$ Gauß: $0 = \int (\vec{\nabla}\cdot\vec{B})\,dV = \oint_{\partial V} \vec{B}\cdot \underbrace{d\vec{a}}_{2\pi\rho\, dl\, \vec{e}_\rho} = 2\pi \int dl\, \rho\, B_\rho(\rho)$

$\Leftrightarrow \rho B_\rho = 0$

$\vec{\nabla}\times\vec{B} = \mu_0 j_0 \vec{e}_z \hookrightarrow B_z = 0 \Rightarrow \boxed{\vec{B}(\vec{r}) = B_\varphi(\rho)\,\vec{e}_\varphi}$

Berechne B_φ: $2\pi\rho B_\varphi(\rho) = \mu_0 \int \vec{j}\cdot d\vec{a}$ (über Stokes: $\vec{\nabla}\times\vec{B} = \mu_0\vec{j}$; $\oint_{\partial F} \vec{B}\cdot d\vec{l} = \mu_0 \int_F \vec{j}\cdot \underbrace{d\vec{f}}_{\rho\, d\rho\, d\varphi\, \vec{e}_z}$)

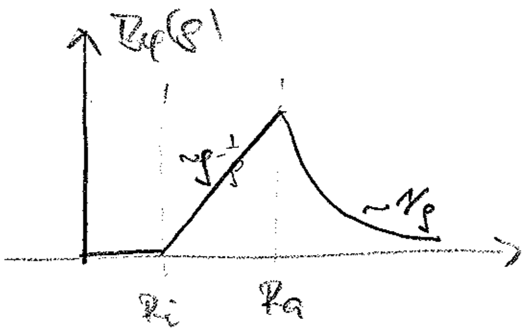
$$= \begin{cases} 0 & \text{für } \rho \le R_i \\ \mu_0 I & \text{für } \rho \ge R_a \\ \mu_0 I \dfrac{\rho^2 - R_i^2}{R_a^2 - R_i^2} & \text{für } R_i < \rho < R_a \end{cases}$$

$$\int_{R_i<\rho<R_a} \vec{j}\cdot d\vec{a} = j_0 \int_0^{2\pi} d\varphi \int^{\rho} d\rho\, \rho = \underbrace{2\pi j_0 (R_a^2 - R_i^2)}_{=I} (\rho^2 - \rho_0^2)$$

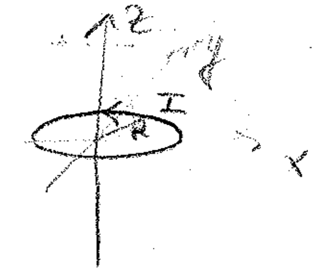
Normierung auf Fläche, j_0: Stromdichte

Integrationskonstante: $\rho_0 \to R_i$ damit $= 0$ für $\rho = R_i$

$$B_\varphi(\rho) = \frac{\mu_0 I}{2\pi\rho} \begin{cases} 0 & , \rho \le R_i \\ \dfrac{\rho^2 - R_i^2}{R_a^2 - R_i^2} & , R_i < \rho < R_a \\ 1 & , \rho \ge R_a \end{cases}$$



③ Kreisförmige Leiterschleife



(a) Biot - Savart:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{r}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

, Feld entlang z-Achse: $\vec{r} \sim \vec{e}_z$

Zylinderkoordinaten: $\vec{r}' = R\,\vec{e}_{\rho'} = R\begin{pmatrix}\cos\varphi'\\ \sin\varphi'\\ 0\end{pmatrix}$

$(\vec{e}_\rho \times \vec{e}_\varphi = \vec{e}_z)$ $\qquad d\vec{r}' = R\,d\varphi'\,\vec{e}_{\varphi'} = R\,d\varphi'\begin{pmatrix}-\sin\varphi'\\ \cos\varphi'\\ 0\end{pmatrix}$

$\vec{r} - \vec{r}' = \begin{pmatrix}-R\cos\varphi'\\ -R\sin\varphi'\\ z\end{pmatrix}$, $\quad d\vec{r}' \times (\vec{r} - \vec{r}') = R\,d\varphi'\,\vec{e}_{\varphi'} \times (-R\,\vec{e}_{\rho'} + z\,\vec{e}_z)$

$= R\,d\varphi'(+R\,\vec{e}_z + z\,\vec{e}_{\rho'})$

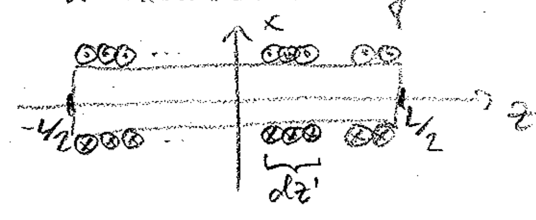
$|\vec{r} - \vec{r}'| = \sqrt{R^2 + z^2}$ $\qquad = R\,d\varphi'\begin{pmatrix}z\cos\varphi'\\ z\sin\varphi'\\ R\end{pmatrix}$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\varphi' \, \frac{R\begin{pmatrix}z\cos\varphi'\\ z\sin\varphi'\\ R\end{pmatrix}}{\sqrt{R^2+z^2}^{\,3}} = \frac{\mu_0 I R}{4\pi} \frac{1}{\sqrt{R^2+z^2}^{\,3}} \begin{pmatrix}0\\0\\2\pi R\end{pmatrix}$$

Integration über volle Periode

$$= \frac{\mu_0 I}{2} \frac{R^2}{\sqrt{R^2+z^2}^{\,3}}\,\vec{e}_z = \begin{cases} \vec{e}_z \dfrac{\mu_0 I}{2} \dfrac{R^2}{R^3} = \vec{e}_z \dfrac{\mu_0 I}{2R}, & z \ll R \\[2mm] \vec{e}_z \dfrac{\mu_0 I}{2} \dfrac{R^2}{z^3}\left(1 - \dfrac{3}{2}\dfrac{R^2}{z^2} + \dots\right), & z \gg R \end{cases}$$

(b) lange, dicht gewickelte Spule (Länge L, Radius R, Windungszahl N)
↳ Aneinanderreihung kreisförmiger Leiterschleifen



$dN = \frac{N}{L}\,dz' = n\,dz'$ (# Windungen im Intervall dz')

↳ Strom $dI = I\,n\,dz'$

Magnetfeld in z-Richtung (aus (a)) $\quad dB_z = \frac{\mu_0}{2}\,dI\,\frac{R^2}{\sqrt{R^2+z^2}^{\,3}}$

$$B_z(z) = \frac{\mu_0 I}{2}\, n R^2 \int_{-L/2}^{L/2} \frac{dz'}{[R^2 + (z-z')^2]^{3/2}} = \frac{\mu_0 I}{2}\, n \left[\frac{L/2 - z}{[R + (z - L/2)^2]^{1/2}} + \frac{L/2 + z}{[R^2 + (z + L/2)^2]^{1/2}}\right]$$

sehr lange Spule: $\lim_{L\to\infty} B_z = \frac{\mu_0 I n}{2} \cdot 2 = \mu_0 I n = \text{const.}$

4 Kraft auf lokale Stromverteilung

$$\vec F = \int d^3r\, \vec j(\vec r)\times(\vec r\cdot\vec\nabla_r)\vec B(\vec r)\Big|_{\vec r=0} + \dots$$

$$\Leftrightarrow \vec F = \frac{1}{2}\int \left[d^3r\,(\vec r\times\vec j(\vec r))\times\vec\nabla_r\right]\times\vec B(\vec r)\Big|_{\vec r=0} + \dots$$

$$\to F_i = \varepsilon_{ijk}\int d^3r\, j_j\, \underbrace{(\vec r\cdot\vec\nabla)\, B_k(0)}_{\vec r\cdot(\vec\nabla B_k(0))} \quad (*) \qquad \vec B(\vec r)\big|_{\vec r=0} = \vec B(0)$$

$$= \varepsilon_{ijk}\,\delta_{mn}\int d^3r\, j_j\, r_m\, \partial_n B_k$$

Nutze Kontinuitätsgleichung $\vec\nabla\cdot\vec j = 0$:

$$\int d^3r\, r_k\, j_l \overset{\vec\nabla\cdot\vec j=0}{=} \int d^3r\, r_k\, \underbrace{\vec\nabla(r_l\,\vec j)}_{\delta_{li}\,j_i + 0} \overset{\text{partiell integrieren}}{=} -\int d^3r\,(\vec\nabla r_k)\, r_l\,\vec j = -\int d^3r\, r_l\, j_k$$

aus

$$\int d^3r\,(\vec a\cdot\vec r)\,\vec j = -\int d^3r\,(\vec a\cdot\vec j)\,\vec r \overset{\text{bac-cab}}{=} -\frac{1}{2}\,\vec a\times\int d^3r\,(\vec r\times\vec j) \quad (**)$$

$$= -\frac{1}{2}\,\underbrace{\varepsilon_{ijk}\, a_j\int d^3r\,\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}}\, r_l\, j_m$$

$$= -\frac{1}{2}\, a_j\int d^3r\,(r_i\, j_j - r_j\, j_i) = \vec a\cdot\int d^3r\,\vec r\, j_i = \int d^3r\,(\vec a\cdot\vec r)\, j_i$$

$$(*) = \varepsilon_{ijk}\int d^3r\, j_j\,\vec r\cdot\vec a_k \quad \text{mit} \quad \vec a_k = \vec\nabla B_k$$

$$\overset{(**)}{=} -\frac{1}{2}\,\varepsilon_{ijk}\Big[\vec\nabla B_k\times\underbrace{\int d^3r\,(\vec r\times\vec j)}_{2\vec m}\Big]_j = -\varepsilon_{ijk}\,[\vec m\times\vec\nabla B_k]_j = -\varepsilon_{ijk}\,[\vec m\times\vec\nabla]_j B_k$$

$$= +\varepsilon_{ijk}\,[\vec m\times\vec\nabla]_j B_k$$

$$\Rightarrow \vec F = (\vec m\times\vec\nabla)\times\vec B(0) = \frac{1}{2}\int d^3r\left[(\vec r\times\vec j)\times\vec\nabla_r\right]\times\vec B(0)$$