(b) homogene Lösung der 1D Wellengleichung:

$$\varphi(x,t) = \mathrm{Re} \int_{-\infty}^{\infty} dk\, a(k) \exp[i(kx-\omega t)] \quad \text{mit } \omega = c|k|$$

Hinweis

$$= \frac{1}{2} \int_{-\infty}^{\infty} dk \left[a(k) e^{i(kx-\omega t)} + a^*(k) e^{-i(kx-\omega t)} \right]$$

$k \to -k$ im 2. Term

$$= \frac{1}{2} \int_{-\infty}^{\infty} dk \left[a(k) e^{i(kx-\omega t)} + a^*(-k) e^{i(kx+\omega t)} \right]$$

mit $\omega = c|k|$ ⟶ nur Funktion von $x \pm ct$ und $\omega = -ck$ für $k<0$

$$= \frac{1}{2} \int_{0}^{\infty} dk \left[a(k) e^{ik(x-ct)} + a^*(-k) e^{ik(x+ct)} \right]$$

$$+ \frac{1}{2} \int_{-\infty}^{0} dk \left[a(k) e^{ik(x+ct)} + a^*(-k) e^{ik(x-ct)} \right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dk \left[\hat{f}(k) e^{ik(x-ct)} + \hat{g}(k) e^{ik(x+ct)} \right]$$

wobei $\hat{f}(k) = \begin{cases} a(k) & \text{für } k>0 \\ a^*(-k) & k \le 0 \end{cases}$, $\hat{g}(k) = \begin{cases} a^*(-k), & k>0 \\ a(k), & k \le 0 \end{cases}$

$\to$ Fouriertrafo: $\varphi(x,t) = \underbrace{\sqrt{\frac{\pi}{2}}}\left[f(x-ct) + g(x+ct) \right]$

F.T. wurde def. als $\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}$

φ reell? $\varphi^* = \frac{1}{2} \int_{-\infty}^{\infty} dk\, \underbrace{\hat{f}^*(k)}_{\hat{f}(-k)} e^{-ik(x-ct)} + \underbrace{\hat{g}^*(k)}_{\hat{g}(-k)} e^{-ik(x+ct)} = \varphi$ ✓

$k \to -k$: $-\int_{+\infty}^{-\infty} dk = \int_{-\infty}^{\infty} dk$

③ Elektromagnetische Kugelwelle

$$\vec{E}(\vec{r},t) = \vec{E}_0(\theta,\varphi)\,\frac{e^{i(kr-\omega t)}}{r}\,,\quad \vec{B}(\vec{r},t) = \vec{B}_0(\theta,\varphi)\,\frac{e^{i(kr-\omega t)}}{r}$$

(a) Vektorpot. in Coulomb-Eichung: $\vec{\nabla}\cdot\vec{A} = 0$

$$\left.\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A}\\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\end{aligned}\right\}$$

quellenfrei: $\vec{\nabla}\cdot\vec{E} = 0 = -\Delta\phi$

Eichfreiheit: wähle $\phi \equiv 0$

(Warum? Eichfunktion $\frac{\partial\lambda}{\partial t} = \phi$

$$\vec{E} = -\vec{\nabla}\phi + \vec{\nabla}\frac{\partial\lambda}{\partial t} - \frac{\partial\vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\lambda = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \text{ inv.})$$

$$\Rightarrow \frac{\partial\vec{A}}{\partial t} = -\vec{E} \Rightarrow \vec{A} = -\int dt\,\vec{E} = +\frac{\vec{E}_0}{i\omega}\,\frac{e^{i(kr-\omega t)}}{r} + \vec{C}_1$$

Konstanten im Potentialen irrelevant: $\vec{C}_1 = 0$

$$\Rightarrow \boxed{\vec{A} = -\frac{i}{\omega}\vec{E}_0(\theta,\varphi)\,\frac{e^{i(kr-\omega t)}}{r}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{e}_r\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,E_{0,\varphi}\right) - \frac{\partial E_{0,\theta}}{\partial\varphi}\right]\frac{-i}{\omega}\,\frac{e^{i(kr-\omega t)}}{r}$$

$$+\,\vec{e}_\theta\left[\frac{1}{r\sin\theta}\,\frac{\partial E_{0,r}}{\partial\varphi}\,\frac{e^{i(kr-\omega t)}}{r} - \frac{E_{0,\varphi}}{r}\underbrace{\frac{\partial}{\partial r}\left(e^{i(kr-\omega t)}\right)}_{ik\,e^{i(kr-\omega t)}}\right]\frac{-i}{\omega}$$

$$+\,\vec{e}_\varphi\,\frac{-i}{\omega r}\left[E_{0,\theta}\frac{\partial}{\partial r}e^{i(kr-\omega t)} - \frac{\partial E_{0,r}}{\partial\theta}\,\frac{e^{i(kr-\omega t)}}{r}\right]$$

Propagation der Kugelwelle in radialer Richtung: $\vec{k} \sim \vec{e}_r$

$$\Rightarrow B_r \overset{!}{=} 0 \Rightarrow \boxed{\frac{\partial}{\partial\theta}\left(\sin\theta\,E_{0,\varphi}\right) - \frac{\partial E_{0,\theta}}{\partial\varphi} = 0}$$

und wegen der Transversalität $B_{0,r} = 0$, $E_{0,r} = 0$

damit bleibt

$$\vec{B} = \vec{e_\vartheta}\left(-E_{0,\varphi}\frac{e^{i(kr-\omega t)}}{r}\frac{k}{\omega} + \vec{e_\varphi}\, E_{0,\vartheta}\frac{k}{\omega}\frac{e^{i(kr-\omega t)}}{r}\right.$$

$$= \frac{k}{\omega}\left(-E_{0,\varphi}\vec{e_\vartheta} + E_{0,\vartheta}\vec{e_\varphi}\right)\frac{e^{i(kr-\omega t)}}{r}$$

(b) $\underbrace{\vec{\nabla}\times\vec{B}}_{\parallel} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \Leftrightarrow \vec{E} = c^2\int dt\, \vec{\nabla}\times\vec{B}$

$$\vec{e_\vartheta}\left[-\frac{1}{r}\frac{k}{\omega}E_{0,\vartheta}\underbrace{\frac{\partial}{\partial r}e^{i(kr-\omega t)}}_{ik\,e^{i(kr-\omega t)}}\right] + \vec{e_\varphi}\left[\frac{1}{r}\frac{k}{\omega}(-E_{0,\varphi})\frac{\partial}{\partial r}e^{i(kr-\omega t)}\right]$$

$$\vec{E} = ik\left[-\omega B_{0,\varphi}\vec{e_\vartheta} + \omega B_{0,\vartheta}\vec{e_\varphi}\right]c^2\underbrace{\int dt\,\frac{e^{i(kr-\omega t)}}{r}}_{-\frac{1}{i\omega}\frac{e^{i(kr-\omega t)}}{r}}$$

$$\begin{pmatrix} B_{0,\vartheta} = -\frac{k}{\omega}E_{0,\varphi} \\ B_{0,\varphi} = \frac{k}{\omega}E_{0,\vartheta} \end{pmatrix} \quad (*)$$

$$\Leftrightarrow \frac{-i\omega}{c^2}\vec{E}_0 = \left(-ikB_{0,\varphi}\vec{e_\vartheta} + ikB_{0,\vartheta}\vec{e_\varphi}\right) \quad (**)$$

(c) Dispersionsrelation $\omega(k)$

aus Koeff.-vgl.: $E_{0,\vartheta} = \frac{+k}{\omega}c^2 B_{0,\varphi}$ aus $(**)$

mit $(*) \to E_{0,\vartheta} = \frac{k}{\omega}\left(+\frac{k}{\omega}\right)c^2 E_{0,\vartheta}$

entsprechend $E_{0,\varphi} = \frac{k}{\omega}c^2 B_{0,\vartheta} = c^2\frac{k}{\omega}\frac{k}{\omega}E_{0,\varphi}$

$$\to c^2\frac{k^2}{\omega^2} = 1 \Leftrightarrow \boxed{\omega^2 = c^2k^2}$$

(d) Poynting-Vektor: $\langle \vec{S} \rangle_T = \frac{1}{2\mu_0} \mathrm{Re}(\vec{E} \times \vec{B}^*)$

$$= \frac{1}{2\mu_0 r^2} \left(E_{0,\theta} \vec{e}_\theta + E_{0,\varphi} \vec{e}_\varphi \right) \times \left(B_{0,\theta} \vec{e}_\theta + B_{0,\varphi} \vec{e}_\varphi \right)$$

$$= \frac{1}{2\mu_0 r^2} \left(E_{0,\theta} B_{0,\varphi} - E_{0,\varphi} B_{0,\theta} \right) \vec{e}_r$$

$$= \frac{1}{2\mu_0 r^2} \left(\frac{k}{\omega} E_{0,\theta}^2 + \frac{k}{\omega} E_{0,\varphi}^2 \right) \vec{e}_r = \frac{1}{2\mu_0 r^2} \underbrace{\frac{k}{\omega}}_{\frac{1}{c}} \underbrace{\left(E_{0,\theta}^2 + E_{0,\varphi}^2 \right)}_{\vec{E}_0^2} \vec{e}_r$$

$$= \frac{1}{2 c r^2} \frac{1}{\mu_0} \underbrace{c^2 \mu_0 \varepsilon_0}_{1} E_0^2 \vec{e}_r$$

$$= \frac{c \varepsilon_0}{2 r^2} E_0^2 \vec{e}_r$$

(e) $\psi(\vec{r}, t) = \frac{e^{i(kr - \omega t)}}{r}$ nach ebenen Wellen entwickeln

F.T.: $\hat{\psi}(\vec{k}, \Omega) = \frac{1}{(2\pi)^2} \int d^3 r \int dt \, e^{-i(\vec{k}\cdot\vec{r} - \Omega t)} \frac{e^{i(kr - \omega t)}}{r}$

$$= \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i(\Omega - \omega)t}}_{\delta(\Omega - \omega)} \frac{1}{2\pi} \underbrace{\int d^3 r \frac{1}{r} e^{i(kr - \vec{k}\cdot\vec{r})}}_{\int_0^\infty r^2 dr \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_{-1}^{1} dx \, e^{i(kr - Krx)}}$$

$$= \delta(\Omega - \omega) \int_0^\infty dr \, r \int_{-1}^{1} dx \, e^{i(kr - Krx)}$$

$$= \delta(\Omega - \omega) \int_0^\infty dr \, r \frac{1}{-iKr} e^{ikr} \left(e^{-iKr} - e^{+iKr} \right) = \frac{i}{K} \delta(\Omega - \omega) \int_0^\infty dr \left(e^{i(k-K)r} - e^{i(k+K)r} \right)$$

$$= \frac{i}{K} \delta(\Omega - \omega) \left[\frac{1}{i(k-K)} e^{i(k-K)r} \Big|_0^\infty - \frac{1}{i(k+K)} e^{i(k+K)r} \Big|_0^\infty \right.$$

bel. kl. pos. Imaginärteil: $k \to k + i\varepsilon$, $\varepsilon > 0$: exp-Dämpfung f. $r \to \infty$

$$\Rightarrow e^{i(k \pm K)r} \to e^{i(k \pm K)r} e^{-\varepsilon r} \Big|_0^{r\to\infty} = 0 - 1$$

$$\Rightarrow \hat{\psi}(\vec{K}, t) = \frac{1}{K}\,\delta(\Omega - \omega)\underbrace{\left[\frac{-1}{k-K} - \frac{-1}{k+K}\right]}_{\frac{-(k+K)+(k-K)}{k^2-K^2} = \frac{-2K}{k^2-K^2}}$$

$$= \frac{2}{K^2 - k^2}\,\delta(\Omega - \omega)$$

Rücktrafo: $\psi(\vec{r}, t) = \frac{1}{(2\pi)^2}\int d^3K \int d\Omega\, \frac{2\,\delta(\Omega-\omega)}{K^2-k^2}\, e^{i(\vec{K}\cdot\vec{r}-\Omega t)}$

$$= \frac{1}{2\pi^2}\int d^3K\, \frac{e^{i(\vec{K}\cdot\vec{r}-\omega t)}}{K^2 - k^2}$$

$\Rightarrow$ Entwicklung nach ebenen Wellen $\left(e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right)$

mit Entw.koeff. $(K^2 - k^2)^{-1}$