① Relativistische Addition von Geschwindigkeiten und Rapiditäten

(a) $K \to K'$

$$\begin{cases} x \to x' = \gamma(x - \beta ct) \\ ct \to ct' = \gamma(ct - \beta x) \end{cases} \quad , \quad \gamma = (1-\beta^2)^{-1/2}, \quad \beta = \frac{v}{c}$$

Hintereinanderausführung von Lorentz-Trafos:

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \Lambda(\beta_2)\,\Lambda(\beta_1) \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\left[\text{wobei } \Lambda(\beta) = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \equiv \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \right]$$

η: „Rapidität"

$$= \gamma_1\gamma_2 \underbrace{\begin{pmatrix} 1 & -\beta_2 \\ -\beta_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\beta_1 \\ -\beta_1 & 1 \end{pmatrix}}_{\begin{pmatrix} 1+\beta_1\beta_2 & -\beta_1-\beta_2 \\ -\beta_1-\beta_2 & 1+\beta_1\beta_2 \end{pmatrix}} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$= \gamma_1\gamma_2(1+\beta_1\beta_2) \begin{pmatrix} 1 & -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \\ -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\equiv \gamma_{1+2} \begin{pmatrix} 1 & -\beta_{1+2} \\ -\beta_{1+2} & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \quad \beta_{1+2} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

$$\gamma_{1+2} = \gamma_1\gamma_2(1+\beta_1\beta_2) = \sqrt{\frac{(1+\beta_1\beta_2)^2}{(1-\beta_1^2)(1-\beta_2^2)}} = \sqrt{\frac{1+2\beta_1\beta_2+\beta_1^2\beta_2^2}{1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2}}$$

$$\left\{ \begin{aligned} &1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2 \\ &= (1+\beta_1\beta_2)^2 - (\beta_1+\beta_2)^2 \end{aligned} \right.$$

$$= \left[1 - \frac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}\right]^{-1/2} = \frac{1}{\sqrt{1-\beta_{1+2}^2}}$$

(b) $$\begin{cases} x' = x\cosh\eta - ct\sinh\eta \\ ct' = ct\cosh\eta - x\sinh\eta \end{cases}$$

$$\left.\begin{aligned} \sinh\eta &= \gamma\beta \\ \cosh\eta &= \gamma \end{aligned}\right\} \Rightarrow \tanh\eta = \beta \Rightarrow \eta = \operatorname{Artanh}\beta$$

Hintereinanderausführung:

$$\Lambda(\underbrace{\eta_1+\eta_2}_{\eta_{1+2}}) = \Lambda(\eta_2)\,\Lambda(\eta_1) = \begin{pmatrix} \cosh\eta_2 & -\sinh\eta_2 \\ -\sinh\eta_2 & \cosh\eta_2 \end{pmatrix}\begin{pmatrix} \cosh\eta_1 & -\sinh\eta_1 \\ -\sinh\eta_1 & \cosh\eta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{ch}\eta_2\,\mathrm{ch}\eta_1 + \mathrm{sh}\eta_1\,\mathrm{sh}\eta_2 & -\mathrm{sh}\eta_1\,\mathrm{ch}\eta_2 - \mathrm{ch}\eta_1\,\mathrm{sh}\eta_2 \\ -\mathrm{ch}\eta_1\,\mathrm{sh}\eta_2 - \mathrm{sh}\eta_1\,\mathrm{ch}\eta_2 & \mathrm{sh}\eta_1\,\mathrm{sh}\eta_2 + \mathrm{ch}\eta_1\,\mathrm{ch}\eta_2 \end{pmatrix}$$

Additionstheoreme für Hyperbelfunktionen $\rightarrow$ $= \begin{pmatrix} \cosh(\eta_1+\eta_2) & -\sinh(\eta_1+\eta_2) \\ -\sinh(\eta_1+\eta_2) & \cosh(\eta_1+\eta_2) \end{pmatrix}$

$$\Rightarrow \eta_{1+2} = \eta_1 + \eta_2$$

„Hintereinanderausführung von Lorentz-Trafos ist selbst auch wieder Lorentz-Trafo"

$\Rightarrow$ Gruppeneigenschaft

② Relativistische Bewegungsgleichungen

Lorentz-Kraft: $\vec{F}_L = q(\vec{E} + \vec{v}\times\vec{B})$

kovariante Verallgemeinerung der Newtonschen Kraft:

$K^\mu = (K^0, \vec{K})^T$, mit $K^0 = \frac{1}{c}\frac{\vec{F}\cdot\vec{v}}{\sqrt{1-\beta^2}}$

und $K^i = \frac{F^i}{\sqrt{1-\beta^2}}$, $\beta^2 = \frac{\vec{v}^2}{c^2}$

suche kovariante Form der Bewegungsgleichungen

Newton: $\frac{dp_i}{dt} = (\vec{F}_L)_i = \frac{K^i}{\gamma}$

def. Eigenzeit τ: $t = \gamma\tau \Rightarrow dt = \gamma\, d\tau$

$\Rightarrow \boxed{\frac{dp_i}{d\tau} = K^i}$

$$\underbrace{\vec{v}\cdot\frac{d\vec{p}}{dt}}_{\frac{cK^0}{\gamma}} = \vec{v}\frac{d}{dt}(\gamma m\vec{v}) = mv^2\frac{d\gamma}{dt} + m\gamma\,\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$= mv^2\frac{d\gamma}{dt} + \frac{1}{2}m\gamma\frac{dv^2}{dt} = \frac{d}{dt}(m\gamma c^2) = c\frac{dp^0}{dt}$$

$$\frac{d\gamma}{dt} = \left(-\frac{1}{2}\right)\gamma^3\left(-\frac{dv^2}{dt}\right) = \frac{1}{2}\gamma^3\frac{1}{c^2}\frac{dv^2}{dt^2}$$

$$\Leftrightarrow \frac{dv^2}{dt} = 2c^2\gamma^{-3}\frac{d\gamma}{dt}$$

$$\Rightarrow \frac{1}{2}m\gamma\frac{dv^2}{dt} = mc^2\gamma^{-2}\frac{d\gamma}{dt}$$

$$= mc^2\left(1-\frac{v^2}{c^2}\right)\frac{d\gamma}{dt} = mc^2\frac{d\gamma}{dt} - mv^2\frac{d\gamma}{dt}$$

$$\Rightarrow \frac{dp^0}{dt} = \frac{K^0}{\gamma} \Leftrightarrow \frac{dp^0}{d\tau} = K^0$$

$$\Rightarrow \boxed{\frac{dp^\mu}{d\tau} = K^\mu}$$ kovariante Form der BGL

Bedeutung?

$$\frac{dp^0}{d\tau} = \frac{\gamma}{c}\vec{F}\cdot\vec{v} = \frac{\gamma}{c}\vec{F}\cdot\frac{d\vec{s}}{dt} \underset{\uparrow}{=} \frac{\gamma}{c}\frac{dE}{dt}$$

$$\frac{dp^0}{d\tau} = K^0$$

sofern $\vec{F}$ = const.
$\vec{F}\cdot\vec{s} \to E$ (Arbeit)

$$\frac{d\vec{p}}{d\tau} = \gamma\vec{F}$$

$$\frac{d\vec{p}}{d\tau} = \vec{K}$$

$$\Rightarrow \frac{dp^\mu}{d\tau} = \gamma\frac{d}{dt}p^\mu = \gamma\Big(\underbrace{\frac{1}{c}\frac{dE}{dt}}_{\text{Leistung}}, \underbrace{\frac{d\vec{p}}{dt}}_{\text{Kraft}}\Big)$$

4-Impuls: $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

$$p^2 = \underbrace{p_\mu p^\mu}_{\text{Skalarprodukt = Lorentz-invariant}} = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$$

Invariante = Masse

Poincaré-Symmetrie: Lorentz-Boost + Shift

Generator von Translationen = Impuls

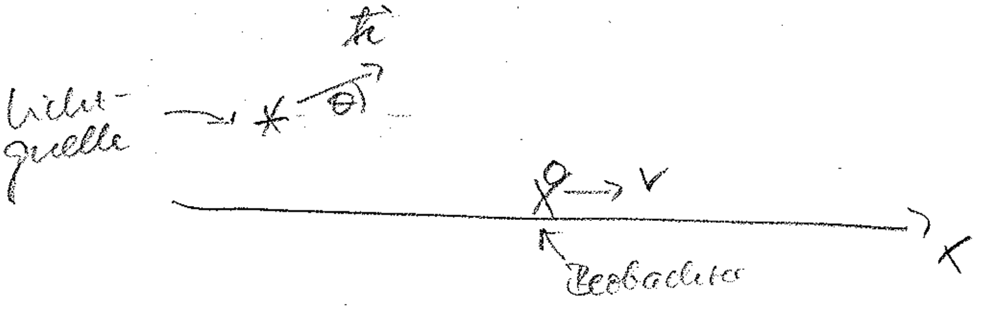
$\leadsto$ Casimir-Invariante = Masse

③ Doppler-Effekt

Monochromatische e.m. Welle:

$$\psi(x^\mu) = A e^{i k \cdot x}$$

$$k \cdot x = k_\mu x^\mu = \omega t - \vec{k} \cdot \vec{x}$$



(a) Beobachter misst Frequenz ω': Transformation in sein Inertialsystem:

$$\begin{pmatrix} \omega'/c \\ k'^1 \\ k'^2 \\ k'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \omega/c \\ k^1 \\ k^2 \\ k^3 \end{pmatrix}, \text{ wobei } k^1 = |\vec{k}| \cos\theta$$

$$\Rightarrow \begin{cases} \omega'/c = \gamma \omega/c - \gamma\beta k^1 \;\leadsto\; \omega' = \gamma(\omega - \underbrace{\beta c k}_{\text{} } \cos\theta) \\ k'^1 = -\gamma\beta \omega/c + \gamma k^1 \end{cases}$$

$|\vec{k}| = \omega/c$ (Dispersionsrelation)

$$\boxed{\omega' = \gamma\omega(1 - \beta\cos\theta)}$$

$$= \gamma(k^1 - \beta k) = \gamma k (\cos\theta - \beta) \quad \circledast$$

$$k^2 = \underbrace{k_\mu k^\mu = k'_\mu k'^\mu}_{\text{Lorentz-inv.}} = \frac{\omega^2}{c^2} - \vec{k}^2 = 0 \quad (\text{lichtartig})$$

$$= \frac{\omega'^2}{c^2} - \vec{k}'^2$$

Der Beobachter sieht die ebene Wellen unter einem Winkel θ': $k'^1 = k' \cos\theta'$, $k' = \frac{\omega'}{c}$

$$\circledast \leadsto \cos\theta' = \frac{\gamma k (\cos\theta - \beta)}{k'} \underset{\substack{k = \omega/c \\ k' = \omega'/c}}{=} \frac{\gamma\omega(\cos\theta - \beta)}{\omega'} \underset{\omega' = \gamma\omega(1-\beta\cos\theta)}{=} \frac{\gamma\omega(\cos\theta - \beta)}{\gamma\omega(1 - \beta\cos\theta)}$$

$$= \boxed{\frac{\cos\theta - \beta}{1 - \beta\cos\theta}}$$

(b) $\omega' = \gamma\omega(1-\beta\cos\theta)$

$\cos\theta' = \dfrac{\cos\theta - \beta}{1-\beta\cos\theta}$

(i) $\theta = 0$: $\omega' = \gamma\omega(1-\beta) = \omega\dfrac{1-\beta}{\sqrt{1-\beta^2}} = \omega\underbrace{\sqrt{\dfrac{1-\beta}{1+\beta}}}_{<1} < \omega$

Beobachter sieht geringere Frequenz

↳ „Rotverschiebung"

(ii) $\theta = \pi$: $\omega' = \gamma\omega(1+\beta) = \omega\dfrac{1+\beta}{\sqrt{1-\beta^2}} = \omega\underbrace{\sqrt{\dfrac{1+\beta}{1-\beta}}}_{>1} > \omega$

Beobachter sieht höhere Frequenz

↳ „Blauverschiebung"

(iii) $\theta = \pm\dfrac{\pi}{2}$: $\omega' = \gamma\omega$ („transversaler Dopplereffekt)

④ Gleichförmig beschleunigtes Teilchen

(a) $\frac{d\vec{p}}{dt} = \vec{F}_0 \quad , \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} \quad , \quad \vec{v}(t) = \begin{pmatrix} v(t) \\ 0 \\ 0 \end{pmatrix}$

(⇒ eindimensionales Problem)

$$\frac{dp}{dt} = \frac{d}{dt}(m\gamma v) = F_0 \overset{\text{integrieren}}{\Rightarrow} \underset{\displaystyle \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}}{\underset{\|}{m\gamma v}} = F_0 t \quad , \quad v(0) = 0$$

(Integrationskonst.)

$$m^2 v^2 = \left(1 - \frac{v^2}{c^2}\right) F_0^2 t^2 \Leftrightarrow \left(m^2 c^2 + F_0^2 t^2\right) v^2 = c^2 F_0^2 t^2$$

$$\Leftrightarrow \quad v(t) = \frac{c F_0 t}{\sqrt{m^2c^2 + F_0^2 t^2}} = \frac{act}{\sqrt{1+a^2t^2}} \quad \text{mit } a = \frac{F_0}{mc}$$

integrieren:

$$x(t) = \int_0^t dt'\, v(t') = \frac{c}{a}\left[\sqrt{1+a^2t^2} - 1\right]$$

(b) Zusammenhang Zeit ⇔ Eigenzeit: $\tau = \tau(t)$

$$\tau = \int_0^t \frac{dt'}{\gamma(t')} \quad \text{wobei } \gamma(t) = \sqrt{1+a^2t^2}$$

$$\left(\gamma = \left(1 - v^2/c^2\right)^{-1/2} = \left(1 - \frac{a^2t^2}{1+a^2t^2}\right)^{-1/2} = \left(\frac{1+a^2t^2 - a^2t^2}{1+a^2t^2}\right)^{-1/2} = \sqrt{1+a^2t^2}\right)$$

$$= \int_0^t \frac{dt'}{\sqrt{1+a^2t'^2}} \overset{\text{Knaou's}}{=} \frac{1}{a} \operatorname{Arsinh}(at)$$

$$\Rightarrow \boxed{t = \frac{1}{a} \sinh(a\tau)}$$

damit ist dann

$$v(\tau) = c \tanh(a\tau)$$

$$x(\tau) = \frac{c}{a}\left[\cosh(a\tau) - 1\right]$$

entwickeln für $v \ll c$: $\underset{\overset{\shortparallel}{\frac{F_0 t}{cm}}}{at} \ll 1$ $= \varepsilon$

$$v(\varepsilon) = c\varepsilon - \frac{c}{2}\varepsilon^3 + \mathcal{O}(\varepsilon^2)$$

$$x(\varepsilon) = \underbrace{\frac{c}{2a}\varepsilon^2}_{\sim \frac{p^2}{2m}} - \frac{c}{8a}\varepsilon^4 + \mathcal{O}(\varepsilon^4)$$