① Galilei- und Lorentztransformation der Wellengleichung

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi(x,t) = 0$$

(a) Galilei-Trafo: $\begin{cases} x \to x' = x - vt \\ t \to t' = t \end{cases}$

(i) φ ist Skalar unter Koordinatentransformationen:

$$\varphi'(x',t') = \varphi(x,t) = \varphi(x'+vt',t')$$
$$= \varphi'(x-vt,t)$$

(ii) $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi(x,t) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi'(x',t') = 0$

[1-D Wellengleichung: $\varphi(x,t) = \varphi(x-ct)$]

$$\Rightarrow \frac{\partial\varphi'}{\partial x} = \frac{\partial\varphi'}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial\varphi'}{\partial x'}, \quad \frac{\partial^2\varphi'}{\partial x^2} = \frac{\partial}{\partial x'}\left(\frac{\partial\varphi'}{\partial x'}\right)\frac{\partial x'}{\partial x} = \frac{\partial^2\varphi'}{\partial x'^2}$$

$$\frac{\partial\varphi'}{\partial t} = \frac{\partial\varphi'}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial\varphi'}{\partial t'}\frac{\partial t'}{\partial t} = -v\frac{\partial\varphi'}{\partial x'} + \frac{\partial\varphi'}{\partial t'}$$

$$\frac{\partial^2\varphi'}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{\partial\varphi'}{\partial x'} + \frac{\partial\varphi'}{\partial t'}\right) = -v\frac{\partial^2\varphi'}{\partial x'^2}\frac{\partial x'}{\partial t} - v\frac{\partial^2\varphi'}{\partial t'\partial x'}\frac{\partial t'}{\partial t}$$

$$+ \frac{\partial^2\varphi'}{\partial x'\partial t'}\frac{\partial x'}{\partial t} + \frac{\partial^2\varphi'}{\partial t'^2}\frac{\partial t'}{\partial t} =$$

$$= v^2\frac{\partial^2\varphi'}{\partial x'^2} - v\frac{\partial^2\varphi'}{\partial t'\partial x'} + (-v)\frac{\partial^2\varphi'}{\partial x'\partial t'} + \frac{\partial^2\varphi'}{\partial t'^2}$$

$$= v^2\frac{\partial^2\varphi'}{\partial x'^2} - 2v\frac{\partial^2\varphi'}{\partial t'\partial x'} + \frac{\partial^2\varphi'}{\partial t'^2}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2\varphi'}{\partial t^2} - \frac{\partial^2\varphi'}{\partial x'^2} = \frac{v^2}{c^2}\frac{\partial^2\varphi'}{\partial x'^2} - 2\frac{v}{c^2}\frac{\partial^2\varphi'}{\partial t'\partial x'} + \frac{1}{c^2}\frac{\partial^2\varphi'}{\partial t'^2} - \frac{\partial^2\varphi'}{\partial x'^2} \doteq 0$$

$$\Leftrightarrow \boxed{\left[\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - 2\frac{v}{c^2}\frac{\partial^2}{\partial t'\partial x'} - \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x'^2}\right]\varphi'(x',t') = 0}$$

(b) Wellengleichung invariant unter Lorentz-Trafos:

$$\begin{cases} x \to x' = \gamma x - \beta\gamma ct \\ ct \to ct' = \gamma ct - \beta\gamma x \end{cases}$$

$$\frac{\partial x'}{\partial x} = \gamma, \quad \frac{\partial x'}{\partial t} = -\beta\gamma c, \quad \frac{\partial t'}{\partial x} = -\frac{\beta\gamma}{c}, \quad \frac{\partial t'}{\partial t} = \gamma$$

$$\frac{\partial \varphi'}{\partial x} = \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\frac{\partial^2 \varphi'}{\partial x^2} = \frac{\partial^2 \varphi'}{\partial x'^2}\left(\frac{\partial x'}{\partial x}\right)^2 + \frac{\partial^2 \varphi'}{\partial x'\partial t'}\frac{\partial x'}{\partial x}\frac{\partial t'}{\partial x}$$
$$+ \frac{\partial^2 \varphi'}{\partial t'\partial x'}\frac{\partial t'}{\partial x}\frac{\partial x'}{\partial x} + \frac{\partial^2 \varphi'}{\partial t'^2}\left(\frac{\partial t'}{\partial x}\right)^2$$
$$= \gamma^2\frac{\partial^2 \varphi'}{\partial x'^2} - \frac{\beta\gamma^2}{c}\frac{\partial^2 \varphi'}{\partial x'\partial t'} - \frac{\beta\gamma^2}{c}\frac{\partial^2 \varphi'}{\partial t'\partial x'} + \frac{\beta^2\gamma^2}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2}$$

$$\frac{\partial \varphi'}{\partial t} = \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial t}$$

$$\frac{\partial^2 \varphi'}{\partial t^2} = \frac{\partial^2 \varphi'}{\partial x'^2}\left(\frac{\partial x'}{\partial t}\right)^2 + \frac{\partial^2 \varphi'}{\partial x'\partial t'}\frac{\partial x'}{\partial t}\frac{\partial t'}{\partial t}$$
$$+ \frac{\partial^2 \varphi'}{\partial x'\partial t'}\frac{\partial t'}{\partial t}\frac{\partial x'}{\partial t} + \frac{\partial^2 \varphi'}{\partial t'^2}\left(\frac{\partial t'}{\partial t}\right)^2$$
$$= \beta^2\gamma^2c^2\frac{\partial^2 \varphi'}{\partial x'^2} - \beta\gamma^2 c\frac{\partial^2 \varphi'}{\partial x'\partial t'} - \beta\gamma^2 c\frac{\partial^2 \varphi'}{\partial t'\partial x'} + \gamma^2\frac{\partial^2 \varphi'}{\partial t'^2}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t^2} - \frac{\partial^2 \varphi'}{\partial x^2} = \beta^2\gamma^2\frac{\partial^2 \varphi'}{\partial x'^2} - 2\frac{\beta\gamma^2}{c}\frac{\partial^2 \varphi'}{\partial x'\partial t'} + \frac{\gamma^2}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2}$$
$$- \gamma^2\frac{\partial^2 \varphi'}{\partial x'^2} + 2\frac{\beta\gamma^2}{c}\frac{\partial^2 \varphi'}{\partial x'\partial t'} - \frac{\beta^2\gamma^2}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2}$$
$$= \underbrace{\gamma^2(\beta^2-1)}_{-1}\frac{\partial^2 \varphi'}{\partial x'^2} + \underbrace{\gamma^2(1-\beta^2)}_{1}\frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2}$$
$$= \frac{1}{c^2}\frac{\partial^2 \varphi'}{\partial t'^2} - \frac{\partial^2 \varphi'}{\partial x'^2} = 0$$

Wellengleichung invariant

(2) Lorentztransformationen des elektromagnetischen Feldes

(a) $F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$

Richtung der z-Achse: $\Lambda^\mu{}_\nu$ hat nur $\Lambda^0{}_3$-Eintrag als Nebendiagonale, $\Lambda^i{}_i = 1$ für $i = 1,2$
($\Lambda^0{}_0 = \gamma$, $\Lambda^0{}_3 = -\beta\gamma$)
$\Lambda^3{}_3 = \gamma$

$-E'^i/c = F'^{0i} = \Lambda^0{}_\rho \Lambda^i{}_\sigma F^{\rho\sigma}$

$= \Lambda^0{}_0 \Lambda^i{}_\sigma F^{0\sigma} + \Lambda^0{}_3 \Lambda^i{}_\sigma F^{3\sigma}$

$\sigma = 0 \to F^{00} = 0$
$\sigma = i$

$-E'_x/c = -\gamma E_x/c - \beta\gamma(-B_y) = -\gamma(E_x/c - \beta B_y)$

$-E'_y/c = -\gamma E_y/c - \beta\gamma(B_x) = -\gamma(E_y/c + \beta B_x)$

$-E'_z/c = -\gamma^2 E_z/c - \beta\gamma(-\beta\gamma E_z/c) = -\gamma^2(E_z/c + \beta^2 E_z/c)$
$= -\gamma^2(1+\beta^2) E_z/c = -E_z/c$

$F'^{ij} = \Lambda^i{}_\rho \Lambda^j{}_\sigma F^{\rho\sigma}$

$B'_x \underset{\substack{i=3\\j=2}}{=} \Lambda^3{}_3 \Lambda^2{}_2 F^{32} + \Lambda^3{}_0 \Lambda^2{}_2 F^{02}$

$= \gamma B_x + \beta\gamma E_y/c = \gamma(B_x + \beta E_y/c)$

$B'_y \underset{\substack{i=1\\j=3}}{=} \Lambda^1{}_1 \Lambda^3{}_0 F^{10} + \Lambda^1{}_1 \Lambda^3{}_3 F^{13}$

$= -\beta\gamma E_x/c + \gamma B_y = \gamma(B_y - \beta E_x/c)$

$B'_z \underset{\substack{i=2\\j=1}}{=} \Lambda^2{}_2 \Lambda^1{}_1 F^{21} = B_z$

(b) duales Feldstärketensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad , \text{ wobei } \varepsilon^{0123} = +1$$

Komponentenweise:

$$\tilde{F}^{01} = \frac{1}{2}\left(\varepsilon^{0123}F_{23} + \varepsilon^{0132}F_{32}\right) = \frac{1}{2}\left(F_{23} - F_{32}\right) = F_{23} = -B_x$$

$$\tilde{F}^{02} = \frac{1}{2}\left(\varepsilon^{0213}F_{13} + \varepsilon^{0231}F_{31}\right) = \frac{1}{2}\left(-F_{13} + F_{31}\right) = -F_{13} = -B_y$$

$$\tilde{F}^{03} = \frac{1}{2}\left(\varepsilon^{0312}F_{12} + \varepsilon^{0321}F_{21}\right) = \frac{1}{2}\left(F_{12} - F_{21}\right) = F_{12} = -B_z$$

$$\tilde{F}^{12} = \frac{1}{2}\left(\varepsilon^{1230}F_{30} + \varepsilon^{1203}F_{03}\right) = \frac{1}{2}\left(-F_{30} + F_{03}\right) = F_{03} = +E_z/c$$

$$\tilde{F}^{13} = \frac{1}{2}\left(\varepsilon^{1320}F_{20} + \varepsilon^{1302}F_{02}\right) = \frac{1}{2}\left(F_{20} - F_{02}\right) = -F_{02} = -E_y/c$$

$$\tilde{F}^{23} = \frac{1}{2}\left(\varepsilon^{2310}F_{10} + \varepsilon^{2301}F_{01}\right) = \frac{1}{2}\left(-F_{10} + F_{01}\right) = F_{01} = +E_x/c$$

$$\Rightarrow \quad \tilde{F}^{\mu\nu} \mathrel{\hat{=}} \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & E_z/c & 0 & E_x/c \\ B_z & -E_y/c & -E_x/c & 0 \end{pmatrix}$$

Achtung! $\quad E^i = -F^{0i} = +F_{0i}$

Vorzeichenwechsel durch Kontraktion mit $g_{\mu\nu}$ ($v^0 = v_0$ aber $v^i = -v_i$)

(c) $F^{\mu\nu} F_{\mu\nu} = F^{\mu\nu} F^{T}_{\nu\mu} = \mathrm{Tr}[F F^T]$

$$= \mathrm{Tr}\left[\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\right]$$

$$= \mathrm{Tr}\left[\begin{pmatrix} -\frac{E_x^2+E_y^2+E_z^2}{c^2} & & & \\ & -\frac{E_x^2}{c^2}+B_z^2+B_y^2 & & \\ & & -\frac{E_y^2}{c^2}+B_z^2+B_x^2 & \\ & & & -\frac{E_z^2}{c^2}+B_x^2+B_y^2 \end{pmatrix}\right]$$

$$= -\frac{2}{c^2}(E_x^2+E_y^2+E_z^2) + 2(B_x^2+B_y^2+B_z^2) = -\frac{2}{c^2}(\vec{E}^2 - c^2\vec{B}^2)$$

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\kappa\lambda} F_{\rho\sigma} F^{\kappa\lambda}$$

$$= -\frac{1}{2}\left(g^{\rho}{}_{\kappa}\, g^{\sigma}{}_{\lambda} - g^{\rho}{}_{\lambda}\, g^{\sigma}{}_{\kappa}\right) F_{\rho\sigma} F^{\kappa\lambda}$$

$$= -\frac{1}{2}\left(F_{\kappa\lambda}F^{\kappa\lambda} - \underbrace{F_{\lambda\kappa}}_{-F_{\kappa\lambda}}F^{\kappa\lambda}\right) = -F_{\kappa\lambda}F^{\kappa\lambda}$$

$$= \frac{2}{c^2}(\vec{E}^2 - c^2\vec{B}^2) \quad (\text{o.a.})$$

$$\tilde{F}^{\mu\nu} F_{\mu\nu} = \mathrm{Tr}[\tilde{F} F^T]$$

$$= \mathrm{Tr}\left[\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\right]$$

$$= \underbrace{-\frac{4}{c}\vec{E}\cdot\vec{B}}_{\text{Pseudoskalar!}} = F^{\mu\nu}\tilde{F}_{\mu\nu} \quad (\text{wegen Zyklizität der Spur})$$

$\varepsilon^{\mu\nu\rho\sigma}$ Pseudotensor:

$$\varepsilon'^{\alpha\beta\gamma\delta} = \det\Lambda\, \varepsilon^{\mu\nu\rho\sigma}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\Lambda^{\gamma}{}_{\rho}\Lambda^{\delta}{}_{\sigma}$$

③ Bewegter Draht

(a) $\rho(\vec r) = \lambda k \frac{\delta(r)}{r}$, k: noch zu bestimmende Konstante

Gesamtladung $Q = \lambda L$ bei Zylinder der Höhe L

$$= \int d^3r\, \rho(\vec r)$$
$$= \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi \int_0^R dr\, r\, \lambda k \frac{\delta(r)}{r}$$

$\int_0^R dr\,\delta(r)\ (\tfrac{1}{2})$

$$= L \cdot 2\pi \cdot \lambda k \left(\frac{1}{2}\right) = L\pi\lambda k \Leftrightarrow k = \frac{1}{\pi}$$

$$\Rightarrow \rho(\vec r) = \frac{\lambda}{\pi r}\delta(r) \quad ; \quad \vec j(\vec r) = 0$$

$\vec E$-Feld: Gaußscher Satz: $\int_{\partial V} d\vec f \cdot \vec E = \frac{1}{\varepsilon_0}\int_V d^3r\, \rho(\vec r)$

$$\Leftrightarrow E \cdot 2\pi r L = \frac{1}{\varepsilon_0}\lambda L$$
$$\Leftrightarrow E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \Rightarrow \vec E(\vec r) = E(r)\underbrace{\vec e_r}_{=\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix}}$$

[wegen $\varphi(\vec r) = \varphi(r)$ ist $\vec E = -\vec\nabla\varphi = -\partial_r \varphi\, \vec e_r$]

(b) Transformation ins Laborsystem

Rücktrafo

$$j^\mu = \begin{pmatrix} c\rho \\ \vec j \end{pmatrix} \to j'^\mu = \Lambda^\mu{}_\nu j^\nu = \begin{pmatrix} \gamma & & & +\beta\gamma \\ & 1 & & \\ & & 1 & \\ +\beta\gamma & & & \gamma \end{pmatrix}\begin{pmatrix} c\frac{\lambda}{\pi r}\delta(r) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \rho'(\vec r') = \gamma \frac{\lambda}{\pi}\frac{\delta(r)}{r}$$
$$j_x' = j_y' = 0 \quad ; \quad j_z' = \beta\gamma \frac{c\lambda}{\pi}\frac{\delta(r)}{r}$$

Transformation in z-Richtung: $r' = r$ ($x' = x$, $y' = y$), $\vec e_{r'} = \vec e_r$

$$\frac{\delta(r')}{r'} = \frac{\delta(r)}{r}$$

$$\rho(\vec r') = \frac{\gamma\lambda}{\pi}\frac{\delta(r')}{r'}$$
$$\vec j'(\vec r') = \vec\beta\, \frac{\gamma\lambda c}{\pi}\frac{\delta(r')}{r'}$$

für die Transformation der Felder kann man die Ergebnisse aus Aufgabe 2 verwenden: $(\vec{B} = 0)$

$E_z' = E_z = 0$

$E_x' = \gamma(E_x - c\beta B_y)$

$E_y' = \gamma(E_y + c\beta B_x)$

$B_z' = B_z = 0$

$B_x' = \gamma(B_x + \beta E_y/c)$

$B_y' = \gamma(B_y - \beta E_x/c)$

$$\Rightarrow \boxed{\vec{E}' = \gamma \vec{E} = \frac{\gamma\lambda}{2\pi\varepsilon_0 r'} \vec{e}_{r'}}$$

$$\Rightarrow \boxed{\vec{B}' = \gamma \frac{\vec{\beta} \times \vec{E}}{c} = \frac{\gamma\beta\lambda}{2\pi\varepsilon_0 r' c} \vec{e}_{\varphi'}} \quad \circledast$$

(b) $\vec{E}$- und $\vec{B}$-Felder direkt aus transformierten $\begin{cases} \rho' = \frac{\gamma\lambda}{\pi} \frac{\delta(r')}{r'} \\ \vec{j}' = \frac{\beta\gamma\lambda c}{\pi} \frac{\delta(r')}{r'} \vec{e}_z \end{cases}$

$\vec{E}$: analog (a) Gauß: $\int_{\partial V'} d\vec{f}' \cdot \vec{E}' = \int_{V'} d^3r' \rho(\vec{r}')$

$$\Rightarrow \boxed{\vec{E}'(\vec{r}') = \frac{\gamma\lambda}{2\pi\varepsilon_0 r'} \vec{e}_{r'}} \qquad \vec{e}_{r'} = \begin{pmatrix} \cos\varphi' \\ \sin\varphi' \\ 0 \end{pmatrix}$$

$\vec{B}$: Stokes: $\underbrace{\oint_{C' = \partial F'} d\vec{r}' \cdot \vec{B}'(\vec{r}')}_{} = \mu_0 \int_{F'} d\vec{f}' \cdot \vec{j}'(\vec{r}')$

$$2\pi r' B_{\varphi'}'(r') = \mu_0 2\pi \int_0^{r'} d\rho' \rho' \left(\beta\gamma\lambda c \frac{\delta(\rho')}{\pi \rho'}\right)$$

Kreis in $x'y'$-Ebene mit Radius r', $\vec{B}' = B_{\varphi'}'(r')\vec{e}_{\varphi'}$

$$= \mu_0 \beta\gamma\lambda c$$

$$\Leftrightarrow B_{\varphi'}'(r') = \mu_0 c \frac{\beta\gamma\lambda}{2\pi r'} \overset{c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}}{=} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\beta\gamma\lambda}{2\pi r'}$$

$$\Rightarrow \boxed{\vec{B}'(\vec{r}') = \underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{\frac{1}{\varepsilon_0 c}} \frac{\gamma\beta\lambda}{2\pi r'} \vec{e}_{\varphi'}}$$

untransformiertes Feld $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r} \vec{e}_r$ $(\vec{e}_{r'} = \vec{e}_r\,!)$

decke mit transformiertem Feld $\circledast$: $\vec{B}' = \frac{\gamma\beta}{c}\left(-E_y \vec{e}_{x'} + E_x \vec{e}_{y'}\right)$

$$= \frac{\gamma\beta\lambda}{2\pi\varepsilon_0 c r'} \underbrace{\left(-\sin\varphi' \vec{e}_{x'} + \cos\varphi' \vec{e}_{y'}\right)}_{\vec{e}_{\varphi'}}$$

$$= \frac{\gamma\beta\lambda}{2\pi\varepsilon_0 c r'} \vec{e}_{\varphi'} \checkmark$$