Klassische Theoretische Physik III
Elektrodynamik WS 16/17

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https://cr.ikp.kit.edu/pargner/teaching/theoc1617/

Addendum to exercise sheet 12 — 01.02.17

This is an addendum to exercise 12 concerning the structure of the Lorentz transformations. In the second part of the exercise sheet, the specific relationship between the inverse of the Lorentz transformation Λ and the transpose of Λ was to be shown. The starting point is the following equation (which gives the condition for the Lorentz transformations to preserve the spacetime distance):

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu} = \eta_{\alpha\beta}.$$

If we want to write this equation as a matrix equation, we need to take the transpose of $\Lambda^{\mu}{}_{\alpha}$. The rows of $\Lambda^{\mu}{}_{\alpha}$ transform with Lorentz index up (this is sometimes called a contravariant transformation), and the column transform with index down (this is sometimes called a covariant transformation). This means that the rows of the transpose of $\Lambda^{\mu}{}_{\alpha}$ have to transform covariantly, and the columns have to transform contravariantly. We thus have

$$\Lambda^{\mu}{}_{\alpha} = \left(\Lambda^T\right)_{\alpha}{}^{\mu}$$

We can then re-write our defining equation for $\Lambda^{\mu}{}_{\alpha}$ as

$$\left(\Lambda^T\right)_{\alpha}{}^{\mu}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}.$$

This looks exactly like the matrix equation $\Lambda^T\eta\Lambda = \eta$. We can now proceed in two ways: we can do our manipulation on the level of the matrix equation, or we can do them on the level of the index notation. Both are consistent, as long as we are careful: going back and forth between index and matrix notation can be slightly tricky.

Now, if we start from $\Lambda^T\eta\Lambda = \eta$ and multiply it by η^{-1} from the left, we get $\eta^{-1}\Lambda^T\eta\Lambda = 1$, where 1 here means the identity matrix. In index notation, this reads

$$\eta^{\rho\alpha}\left(\Lambda^T\right)_{\alpha}{}^{\mu}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta^{\rho\alpha}\eta_{\alpha\beta} = \delta^{\rho}{}_{\beta}.$$

Here δ^{ρ}_{β} is the Kronecker delta function, which gives us the identity matrix in index notation. From this equation, it is easy to identify the inverse, which is just $\Lambda^{-1} = \eta^{-1}\Lambda^T\eta$, or

$$\left(\Lambda^{-1}\right)^{\rho}{}_{\nu} = \eta^{\rho\alpha}\left(\Lambda^T\right)_{\alpha}{}^{\mu}\eta_{\mu\nu}.$$

We can manipulate this a bit further if we want to:

$$\left(\Lambda^{-1}\right)^{\rho}{}_{\nu} = \eta^{\rho\alpha}\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu} = \Lambda_{\nu}{}^{\rho}$$

The object $\Lambda_{\nu}{}^{\rho}$ has rows that transform covariantly and columns that transform contravariantly, unlike $\Lambda^{\nu}{}_{\rho}$ which has rows that transform contravariantly and columns that transform covariantly.

Let us finish by discussing a possible point of confusion. If we use η to raise and lower the indices on the right hand side of $\Lambda^{-1} = \eta^{-1}\Lambda^T\eta$, we get

$$\left(\Lambda^{-1}\right)^{\rho}{}_{\nu} = \left(\Lambda^T\right)^{\rho}{}_{\nu}$$

From this, one would like to conclude that $\Lambda^{-1} = \Lambda^T$. *But this is wrong!* When we originally introduced the transpose of Λ, we did so in the form $\left(\Lambda^T\right)_{\alpha}{}^{\mu}$. This is the specific Lorentz index structure we have to use when we want to read of matrix components of the transpose of Λ. Using this, we get the proper relationship $\Lambda^{-1} = \eta^{-1}\Lambda^T\eta$.