Blatt 4 Tobias Abgabe

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/1a)

$$Q_i = \sum_{j=1}^{2} C_{ij} \phi_j \qquad \text{Sei } \underline{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \qquad \vec{Q} = \begin{pmatrix} Q \\ -Q \end{pmatrix}$$

$$\phi_i = \sum_{j=1}^{2} C_{ij}^{-1} Q_j$$

Da zwei Leiter vorhanden sind ist C_{ij} eine 2×2 Matrix

Für die skalare Kapazität gilt:

$$\bar{C} = \frac{Q}{U} = \frac{Q}{\phi_1 - \phi_2} \; *$$

$$\phi_i = C_{ij} Q_j \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \vec{\phi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \underbrace{\underline{C}^{-1}}_{\text{Matrix}} \vec{Q}$$

$$\phi_1 - \phi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\det \underline{C}} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \begin{pmatrix} Q \\ -Q \end{pmatrix}$$

$$= \frac{1}{\det \underline{C}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} Q \begin{pmatrix} C_{22} + C_{12} \\ -C_{21} - C_{11} \end{pmatrix}$$

$$= Q \frac{C_{22} + C_{12} + C_{21} + C_{11}}{\det \underline{C}}$$

$$\Rightarrow \frac{Q}{\phi_1 - \phi_2} = \frac{C_{11} C_{22} - C_{12} C_{21}}{C_{22} + C_{12} + C_{21} + C_{11}} \overset{*}{=} \bar{C} \quad ✓$$

b)

$$C = \varepsilon_0 \varepsilon_r \frac{A}{d} \overset{\text{in cgs}}{\Rightarrow} [C] = 1 \cdot 1 \cdot \frac{\text{cm}^2}{\text{cm}} = \text{cm} \quad ✓$$

c) $\vec{E}$ und ϕ sind aus Blatt 3 bekannt:

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0 r} \vec{e}_r \quad \text{falls } a \le r < b$$

$$\vec{E} = 0 \quad \text{sonst}$$

φ: Fall $b < r$: $\varphi(r) = 0$

Fall $a < r \le b$: $\varphi(r) = -\frac{Q}{2\pi\varepsilon_0 l} [\ln(r')]_b^r = \frac{Q}{2\pi\varepsilon_0 l} \ln\left(\frac{b}{r}\right)$

Fall $r \le a$: $\varphi(r) = -\frac{Q}{2\pi\varepsilon_0 l} [\ln(r')]_b^a = \frac{Q}{2\pi\varepsilon_0 l} \ln\left(\frac{b}{a}\right)$

c).1

$$W = \frac{\varepsilon_0}{2} \int_V \vec{E}^2 dV' = \frac{\varepsilon_0}{2} \int_0^l \int_0^{2\pi} \int_0^{\infty} \left(\frac{Q}{l 2\pi\varepsilon_0 r'}\right)^2 \Theta(r'-a)\, \Theta(b-r')\, r' dr' d\varphi dz$$

$$= \frac{\varepsilon_0}{2l} \int_a^b \frac{Q^2}{2\pi\varepsilon_0^2 r'} dr' = \frac{Q^2}{4\pi\varepsilon_0 l} [\ln(r')]_a^b = \frac{Q^2}{4\pi\varepsilon_0 l} \ln\left(\frac{b}{a}\right)$$

$$W = \frac{1}{2} \int_V \rho \varphi \, dV' = \frac{1}{2} \int_0^l \int_0^{2\pi} \int_0^{\infty} (\sigma_a \delta(r'-a) + \sigma_b \delta(r'-b))\, \varphi(r') r' dr' d\varphi \qquad \text{Mit } \sigma_a = \frac{Q}{2\pi l a} \text{ bzw } \sigma_b = \frac{-Q}{2\pi l b}$$

$$= \pi l \left(\frac{Q}{2\pi l a} \varphi(a)\, a - \frac{Q}{2\pi l b} \varphi(b)\, b \right)$$

$$= \frac{Q^2}{4\pi\varepsilon_0 L} \ln\left(\frac{b}{a}\right) - 0 \quad ✓$$

c).2 $W = \frac{1}{2} C U^2$

$$\Rightarrow C = \frac{2W}{U^2} = \frac{Q^2}{2\pi\varepsilon_0 l} \ln\left(\frac{b}{a}\right) \cdot \left(\frac{Q}{2\pi\varepsilon_0 l} \ln\left(\frac{b}{a}\right)\right)^{-2}$$

$$= 2\pi\varepsilon_0 l \frac{1}{\ln\left(\frac{b}{a}\right)} \quad ✓$$

c).3

$$C = 2\pi\varepsilon_0 l \frac{1}{\ln(\frac{b}{a})}$$

$$\ln\left(\frac{b}{a}\right) = \frac{2\pi\varepsilon_0 l}{C}$$

$$b = a\, e^{\left(\frac{2\pi\varepsilon_0 l}{C}\right)}$$

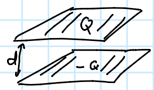
$$= 0{,}0005\,m \quad \text{für} \quad c = 3\cdot 10^{11}$$

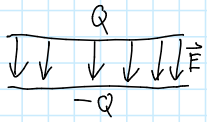
oder $0{,}00319\,m \mathrel{\hat{=}} 3{,}19\,mm$ für $c = 3\cdot 10^{-11}$ ✓
falls das gemeint war

ja !!

d).1

$$\oint_F \vec{E}\, d\vec{f} = \frac{Q}{\varepsilon_0}$$





$$E\cdot A + 0 A = \frac{Q}{\varepsilon_0}$$

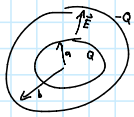
$$E = \frac{Q}{A\varepsilon_0}$$

$$\Delta\phi = \left| -\int_0^d E \; dr' \right| = E d = \frac{Qd}{A\varepsilon_0}$$

$$C = \frac{Q}{\Delta\phi} = \varepsilon_0 \frac{A}{d} \quad ✓$$

d).2 $\quad a < r < b$

$$\int_A E\, dA' = \frac{Q}{\varepsilon_0}$$



$$4\pi r^2 \; E = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\Delta\phi = \left| -\int_a^b E\, dr' \right| = \left| \int_a^b \frac{Q}{4\pi\varepsilon_0 r'^2} dr' \right|$$

$$= \left| \frac{Q}{4\pi\varepsilon_0} \left[-\frac{1}{r'} \right]_a^b \right| = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$= \frac{Q}{4\pi\varepsilon_0} \; \frac{b-a}{ab}$$

$$C = \frac{Q}{\Delta\phi} = 4\pi\varepsilon_0 \frac{ab}{b-a} \quad ✓$$

12 a)

$$\int_V dy^3 \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) = \oint_S da \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right]$$

$$\int_V dy^3 \left(G(x,y) \left(-\frac{1}{\varepsilon_0} \delta(x'-y)\right) - G(x',y) \left(-\frac{1}{\varepsilon_0} \delta(x-y)\right) \right) = \oint_S da \left[G(x,y) \frac{\partial G(x',y)}{\partial n} - G(x',y) \frac{\partial G(x,y)}{\partial n} \right]$$

$$-\frac{1}{\varepsilon_0} G(x,x') + \frac{1}{\varepsilon_0} G(x',x) = \oint_S da \left[G(x,y) \frac{\partial G(x',y)}{\partial n} - G(x',y) \frac{\partial G(x,y)}{\partial n} \right]$$

$$G(x,x') - G(x',x) = -\varepsilon_0 \oint_S da \left[G(x,y) \frac{\partial G(x',y)}{\partial n} - G(x',y) \frac{\partial G(x,y)}{\partial n} \right]$$

b)

Aus der Definition von G folgt:

$$\nabla^2_{x'} G_D(x,x') = -\frac{1}{\varepsilon_0}\delta(x-x') \qquad \delta(r-r') = -\frac{1}{4\pi}\Delta\frac{1}{|r-r'|}$$

$$\Delta G_D(x,x') = \frac{1}{4\pi\varepsilon_0}\Delta\frac{1}{|x-x'|}$$

$$G_D(x,x') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|x-x'|} + f(x,x') \qquad \text{Mit beliebigem } f: \Delta f(x,x') = 0 \quad \text{in } V$$

Dirichlet Randbedingung bedeutet $\varphi(x')$ ist auf S gegeben.

Man wähle $f(x,x')$ so, dass $G_D(x,x') = 0 \quad \forall\, x' \in S$

Dann gilt nach a):

$$G_D(x,x') - G_D(x',x) = -\varepsilon_0 \oint_S da \left[\underbrace{G_D(x,y)}_{=0\,\forall y\in S}\frac{\partial G(x',y)}{\partial n} - \underbrace{G(x',y)}_{=0\,\forall y\in S}\frac{\partial G(x,y)}{\partial n}\right]$$

$$= -\varepsilon_0 [0 - 0]$$

$$= 0$$

Eine Differenz von 0 bedeutet, dass $G_D(x,x') = G_D(x,x')$

$\Rightarrow$ Symmetrie

c)

$$G_N(x,x') - G_N(x',x) = -\varepsilon_0 \oint_S da \left[G_N(x,y)\frac{\partial G_N(x',y)}{\partial n} - G(x',y)\frac{\partial G_N(x,y)}{\partial n}\right]$$

$$= -\varepsilon_0 \oint_S da \left[-G_N(x,y)\frac{1}{\varepsilon_0 S} + G(x',y)\frac{1}{\varepsilon_0 S}\right]$$

$$= \frac{1}{S}\oint_S da \underbrace{\left[G_N(x,y) - G_N(x',y)\right]}_{\text{im Allgemeinen ungleich } 0}$$

$\Rightarrow$ keine zwingende Symmetrie von $G_N(x,x')$ und $G_N(x',x)$

$$H(x,x') = G_N(x,x') - F(x)$$

$$\phi' = G_N(x,y) - F(x) = H(x,y) \qquad \psi' = G_N(x',y) - F(x') = H(x',y)$$

Damit $G_N(x,x') - F(x)$ noch eine die Poisson Gleichung löst, muss $\Delta(-F(x)) = 0$ gelten. (siehe b))
Außerdem ist $F(x)$ unabhängig von y und es handelt sich um $\Delta_y F(x) = 0$

$$\int_V dy^3(\phi'\nabla^2\psi' - \psi'\nabla^2\phi') = \oint_S da\left[\phi'\frac{\partial\psi'}{\partial n} - \psi'\frac{\partial\phi'}{\partial n}\right]$$

$$\int_V dy^3\left(\phi'\left(-\frac{1}{\varepsilon_0}\delta(x'-y) - \Delta F(x')\right) - \psi'\left(-\frac{1}{\varepsilon_0}\delta(x-y) - \Delta F(x)\right)\right) = \dots$$

$$-\frac{1}{\varepsilon_0}H(x,x') + \frac{1}{\varepsilon_0}H(x',x) + \int_V dy^3\left(\Delta F(x)\psi' - \Delta F(x)\,\phi'\right) = \dots$$

$$-\frac{1}{\varepsilon_0}\left(H(x,x') - H(x',x)\right) = \oint_S da\left[\phi'\frac{\partial(G_N(x',y) - F(x'))}{\partial n} - \psi'\frac{\partial(G_N(x,y) - F(x))}{\partial n}\right]$$

$$= \oint_S da\left[\psi'\left(\frac{1}{S\varepsilon_0} + \frac{\partial F(x)}{\partial n}\right) - \phi'\left(\frac{\partial F(x')}{\partial n} - \frac{1}{S\varepsilon_0}\right)\right] \qquad \text{Mit } da\cdot\vec{n} = d\vec{f}$$

$$-\frac{1}{\varepsilon_0}\left(H(x,x') - H(x',x)\right) = -\frac{1}{S\varepsilon_0}\oint_S da\left[\psi' - \phi'\right] + \oint_S d\vec{f}\left[\psi'\overset{=0}{\vec{\Delta} F(x)} - \phi'\overset{=0}{\vec{\Delta} F(x')}\right]$$

$$H(x,x') - H(x',x) = \frac{1}{S}\oint_S da\left[\psi' - \phi'\right]$$

$$= \frac{1}{S}\oint_S da\left[G(x',y) - F(x') - G(x,y) + F(x)\right]$$

$$= \frac{1}{S}\oint da\left[G(x',y) - G(x,y)\right] + \frac{1}{S}\oint_S df\left[F(x) - F(x')\right] \qquad \Big|\ \oint_S d\vec{f} = S$$

$$= \frac{1}{S}\oint da\left[G(x',y) - G(x,y)\right] + F(x) - F(x')$$

$$= \frac{1}{S}\oint da\left[G(x',y) - G(x,y)\right] + \frac{1}{S}\oint_S da_y\left[G_N(x,y) - G_N(x',y)\right]$$

$$= \frac{1}{S}\oint da\left[G(x',y) - G(x,y)\right] - \frac{1}{S}\oint da\left[G(x',y) - G(x,y)\right]$$

$$= 0 \qquad \Rightarrow \text{Symmetrie von } H(x,x') = G_N(x,x') - F(x)$$

$$= \frac{1}{5}\oint da\,[G(x',y) - G(x,y)] - \frac{1}{5}\oint da\,[G(x',y) - G(x,y)]$$
$$= 0 \quad \Rightarrow \text{ Symmetrie von } H(x,x') = G_N(x,x') - F(x)$$

d)

Aus Vorlesung:

$$\varphi_G(x) = \int_V dx'^3 \rho(x')\, G(x,x') - \varepsilon_0 \oint_S da \left[\varphi(x') \frac{\partial G(x,x')}{\partial n} - G(x,x') \frac{\partial \varphi(x')}{\partial n}\right]$$

ersetze $G(x,x')$ mit $H(x,x') = G(x,x') - F(x)$. Dann gilt für $\varphi(x) = \varphi_H(x)$:

$$\varphi_G(x) = \int_V dx'^3 \rho(x')\, H(x,x') - \varepsilon_0 \oint_S da \left[\varphi(x') \frac{\partial H(x,x')}{\partial n} - H(x,x') \frac{\partial \varphi(x')}{\partial n}\right]$$

φ_G ist die Summe der folgenden 3 Integrale:

1. Integral: $\int_V dx'^3 \rho(x')\, H(x,x') = \int_V dx'^3 \rho(x')\, G(x,x') - \int_V dx'^3 \rho(x')\, F(x)$

2. Integral: $-\varepsilon_0 \oint_S da\, \varphi(x') \frac{\partial H(x,x')}{\partial n} = -\varepsilon_0 \oint_S da\, \varphi(x') \left[\frac{\partial G(x,x')}{\partial n} - \overbrace{\frac{\partial F(x)}{\partial n}}^{=0}\right]$
$$= -\varepsilon_0 \oint_S da\, \varphi(x') \frac{\partial G(x,x')}{\partial n}.$$

3. Integral: $\varepsilon_0 \oint_S da\, H(x,x') \frac{\partial \varphi(x')}{\partial n}$
$$= -\varepsilon_0 \oint_S da\, F(x) \frac{\partial \varphi(x')}{\partial n} + \varepsilon_0 \oint_S da\, G(x,x') \frac{\partial \varphi(x')}{\partial n}$$
$$= -\varepsilon_0 \oint_S d\vec{f}\, F(x)\, \nabla \varphi(x') + \varepsilon_0 \oint_S da\, G(x,x') \frac{\partial \varphi(x')}{\partial n}$$
$$\overset{\text{Gauß}}{=} -\varepsilon_0 \int_V dx'^3 F(x)\, \Delta \varphi(x') + \varepsilon_0 \oint_S da\, G(x,x') \frac{\partial \varphi(x')}{\partial n}$$
$$= \int_V dx'^3 F(x)\, \rho(x') + \varepsilon_0 \oint_S da\, G(x,x') \frac{\partial \varphi(x')}{\partial n}$$

Die Summe aller drei Integrale ergibt:

$$\varphi_H(x) = \int_V dx'^3 \rho(x')\, G(x,x') - \int_V dx'^3 \rho(x')\, F(x) - \varepsilon_0 \oint_S da\, \varphi(x') \frac{\partial G(x,x')}{\partial n} + \int_V dx'^3 F(x,x') \rho(x') + \varepsilon_0 \oint_S da\, G(x) \frac{\partial \varphi(x')}{\partial n}$$
$$= \varphi_G(x)$$

Also bleibt das Potential unverändert ✓

3a)

$$W = \frac{1}{2} C \Phi_0^2 = \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} E^2 dx^3$$ Näherung: Äußeres E-Feld kann vernachlässigt werden

$$= \frac{\varepsilon_0}{2} \int_V E^2 dx^3$$
$$= \frac{\varepsilon_0}{2} \int_V (\nabla\phi)^2 dx^3$$

$$\Rightarrow C = \frac{\varepsilon_0}{\phi_0^2} \int_V (\nabla\phi)^2 dx^3$$

Man kann nun $\phi_0^2 = 1$ setzen und erhält die Annahme, allerdings stimmen dann die Einheiten nicht mehr. ✓

b) $\psi = \phi + \delta\psi$

$$\varepsilon_0 \int_V |\nabla\psi|^2 d^3x$$
$$= \varepsilon_0 \int_V |\nabla\phi|^2 d^3x + 2\varepsilon_0 \int_V \nabla\phi\, \nabla\delta\psi\, d^3x + \varepsilon_0 \int_V |\nabla\delta\psi|^2 d^3x \quad | \text{1. Green}$$
$$= C \underbrace{- 2\varepsilon_0 \int_V (\Delta\phi)\, \delta\psi\, d^3x}_{=0} + \underbrace{2\int_S \delta\psi \frac{\partial\phi}{\partial n} da}_{=0} + \underbrace{\varepsilon_0 \int_V |\nabla\delta\psi|^2 d^3x}_{\geq 0} \geq C \quad \blacksquare$$ ✓