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$$\vec\nabla \vec A = \vec\nabla (\vec A_0 e^{i(k_x x - \omega t)})$$

$$= A_x \underbrace{i k_x e^{ikx}}_{\neq 0} \stackrel{!}{=} 0$$

$$\Rightarrow A_{0x} = 0$$

$\vec A_0$ liegt also in der yz-Ebene ✓

a)

$$\vec A_1 = \vec A_{01} e^{i(k_x x - \omega t)} \qquad \vec A_2 = \vec A_{02} e^{i(k_x x - \omega t)}$$

$$\vec E = \underbrace{-\vec\nabla\phi}_{\substack{=0 \\ \text{keine Ladung}}} - \partial_t \vec A = -\partial_t \vec A$$

$$\Rightarrow \vec E_1 = A_1 i\omega e^{i(k_x x - \omega t)} \vec e_y$$

$$\vec E_2 = A_2 i\omega e^{i(k_x x - \omega t + \delta)} \vec e_z$$

Fall $\delta = 0$:

$$\vec E_1 + \vec E_2 = \begin{pmatrix} 0 \\ A_1 \\ A_2 \end{pmatrix} i\omega e^{i(k_x x - \omega t)} = \begin{pmatrix} 0 \\ A_1 \\ A_2 \end{pmatrix} \omega e^{i(k_x x - \omega t + \frac{\pi}{2})}$$ (✓) eig: $\vec E = Re[\ldots]$ -5

Fall $\delta = \pi$:

$$\vec E_1 + \vec E_2 = \begin{pmatrix} 0 \\ A_1 \\ -A_2 \end{pmatrix} \omega i e^{i(k_x x - \omega t)} = \begin{pmatrix} 0 \\ A_1 \\ -A_2 \end{pmatrix} \omega e^{i(k_x x - \omega t + \frac{\pi}{2})}$$ (✓)

b)

$$\delta = \frac{\pi}{2}$$

$$\vec E_1 + \vec E_2 = \begin{pmatrix} 0 \\ iA_1 \\ -A_1 \end{pmatrix} \omega e^{i(k_x x - \omega t)}$$ (✓)

(c))

$$\vec E_1 = E_0 [\cos(kz - \omega t + \phi_0) e_x + \sin(kz - \omega t + \phi_0) \vec e_y]$$

$$\vec E_2 = E_0 [\cos(kz - \omega t) \vec e_x - \sin(kz - \omega t) \vec e_y]$$

$$\vec E_1 + \vec E_2 = E_0 \left[2\cos\left(\frac{2kz - 2\omega t + \phi_0}{2}\right)\cos\left(\frac{\phi_0}{2}\right)\vec e_x + 2\sin\left(\frac{\phi_0}{2}\right)\cos\left(\frac{2kz - 2\omega t + \phi_0}{2}\right)\vec e_y\right]$$

$$= 2E_0 \cos\left(\frac{2kz - 2\omega t + \phi_0}{2}\right)\left[\sin\left(\frac{\phi_0}{2} + \frac{\pi}{2}\right)\vec e_x + \sin\left(\frac{\phi_0}{2}\right)\vec e_y\right]$$

$$= \begin{pmatrix} \sin(\frac{\phi_0}{2} + \frac{\pi}{2}) \\ \sin(\frac{\phi_0}{2}) \\ 0 \end{pmatrix} \cdot 2E_0 \cos\left(kz - \omega t + \frac{\phi_0}{2}\right)$$ lineare Polarisation

d)

$$\vec{E}_1 = \begin{pmatrix} E_1 \\ 0 \\ 0 \end{pmatrix} \cos(kz - \omega t)$$

$$\vec{E}_2 = \begin{pmatrix} \cos(kz-\omega t) \\ -\sin(kz-\omega t) \\ 0 \end{pmatrix}$$

$$\vec{E}_1 + \vec{E}_2 = \begin{pmatrix} 2\cos(kz-\omega t) \\ -\sin(kz-\omega t) \\ 0 \end{pmatrix}$$ Elliptische Polarisation

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e)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}\left[A \frac{\sin^2\theta}{r}\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\right]\vec{e}_r$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\left[A\sin\theta\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\right]\vec{e}_\theta$$
$$= \frac{2A}{r^2}\cos\theta\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\vec{e}_r$$
$$- A\frac{\sin\theta}{r}\left[-k\sin(kr-\omega t) + \frac{1}{kr^2}\sin(kr-\omega t) - \frac{1}{r}\cos(kr-\omega t)\right]\vec{e}_\theta$$

$$\vec{B} = -\int \vec{\nabla}\times\vec{E}\, dt$$
$$= \frac{2A}{r^2}\cos\theta\left(\frac{1}{\omega}\sin(kr-\omega t) + \frac{1}{k\omega r}\cos(kr-\omega t)\right)\vec{e}_r$$
$$+ A\frac{\sin\theta}{r}\left(-\frac{k}{\omega}\cos(kr-\omega t) + \frac{1}{k\omega r^2}\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t)\right)\vec{e}_\theta$$
$$+ \vec{B}_0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[2A\cos\theta\frac{1}{\omega}\left(\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right)\right]$$
$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[A\frac{\sin^2\theta}{\omega r}\left(\left(\frac{1}{kr^2} - k\right)\cos(kr-\omega t) + \frac{1}{r}\sin(kr-\omega t)\right)\right]$$
$$+ \vec{\nabla}\vec{B}_0$$
$$= \frac{2A\cos\theta}{r^2\omega}\left(k\cos(kr-\omega t) - \frac{1}{r}\sin(kr-\omega t) - \frac{1}{kr^2}\cos(kr-\omega t)\right)$$
$$+ \frac{A}{\omega r^2\sin\theta}\cdot 2\sin\theta\cos\theta\left(\left(\frac{1}{kr^2} - k\right)\cos(kr-\omega t) + \frac{1}{r}\sin(kr-\omega t)\right)$$
$$+ \vec{\nabla}\vec{B}_0$$
$$= \vec{\nabla}\vec{B}_0$$

Wähle $\vec{B}_0$ sodass $\vec{\nabla}\vec{B}_0 = 0$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{k^2}{\omega^2}\frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial \vec{E}}{\partial t} = A\frac{\sin\theta}{r}\left[\omega\sin(kr-\omega t) + \frac{\omega}{kr}\cos(kr-\omega t)\right]\vec{e}_\varphi$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(A\sin\theta\left(\left(\frac{1}{k\omega r^2}-\frac{k}{\omega}\right)\cos(kr-\omega t) + \frac{1}{r\omega}\sin(kr-\omega t)\right)\right)\right.$$
$$\left. + \frac{2A}{r^2}\sin\theta\left(\frac{1}{\omega}\sin(kr-\omega t) + \frac{1}{k\omega r}\cos(kr-\omega t)\right)\right]\vec{e}_\varphi$$
$$+ \vec{\nabla}\times\vec{B}_0$$
$$= \frac{1}{r}\left[A\sin\theta\left\{-\frac{2}{k\omega r^3}\cos(kr-\omega t) - k\left(\frac{1}{k\omega r^2}-\frac{k}{\omega}\right)\sin(kr-\omega t)\right.\right.$$
$$\left. - \frac{1}{r^2\omega}\sin(kr-\omega t) + \frac{k}{r\omega}\cos(kr-\omega t)\right\}$$
$$\left. + \frac{2A}{r^2}\sin\theta\left(\frac{1}{\omega}\sin(kr-\omega t) + \frac{1}{k\omega r}\cos(kr-\omega t)\right)\right]\vec{e}_\varphi$$
$$+ \vec{\nabla}\times\vec{B}$$
$$= \frac{A\sin\theta}{r}\left[\frac{k^2}{\omega}\sin(kr-\omega t) + \frac{k}{r\omega}\cos(kr-\omega t)\right] + \vec{\nabla}\times\vec{B}$$
$$= \frac{k^2}{\omega^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}_0$$

Die Maxwell Gleichungen sind also erfüllt wenn

$\vec{\nabla}\vec{B}_0 = \vec{\nabla}\times\vec{B}_0 = 0$ ($\Rightarrow \vec{B}_0$ ist ein konstanter Vektor)

b)

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \qquad \text{wähle } \vec{B}_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} ✓$$

$$= \frac{1}{\mu_0}\frac{A\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\cdot$$
$$\left\{\frac{2A\cos\theta}{\omega r^2}\left[\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right]\vec{e}_\varphi\times\vec{e}_r\right.$$
$$\left. + \frac{A\sin\theta}{\omega r}\left[\left(\frac{1}{kr^2}-k\right)\cos(kr-\omega t) + \frac{1}{r}\sin(kr-\omega t)\right]\vec{e}_\varphi\times\vec{e}_\theta\right\}$$

$$= \frac{2A^2\sin\theta\cos\theta}{\mu_0\omega r^3}\left[\sin\cdot\cos + \frac{1}{kr}\cos^2 - \frac{1}{kr}\sin^2 - \frac{1}{k^2r^2}\sin\cdot\cos\right]\vec{e}_\theta$$
$$+ \frac{A^2\sin^2\theta}{\mu_0\omega r^2}\left[\left(\frac{1}{kr^2}-k\right)\cos^2 + \frac{1}{r}\sin\cdot\cos - \left(\frac{1}{k^2r^3}-\frac{1}{r}\right)\sin\cdot\cos - \frac{1}{k^2r^2}\sin^2\right](-\vec{e}_r)$$

$$T = \frac{\lambda}{c} = \frac{2\pi}{k}\frac{k}{\omega} = \frac{2\pi}{\omega}$$

$$\langle\sin^2(kr-\omega t)\rangle = \langle\cos^2(kr-\omega t)\rangle = \frac{1}{\left(\frac{2\pi}{\omega}\right)}\int_0^{\frac{2\pi}{\omega}}\sin^2(kr-\omega t)\,dt \qquad \Big|\ \text{Phönix}$$
$$= \frac{\omega}{2\pi}\left[\frac{\omega t - \frac{\sin(2(\omega t-kr))}{2}}{2\omega}\right]_0^{\frac{2\pi}{\omega}}$$
$$= \frac{\omega}{2\pi}\left[\left(\frac{\pi}{\omega} - \frac{1}{\omega}\sin(2(2\pi-kr)\right) + \frac{1}{\omega}\sin(-kr)\right]$$
$$= \frac{1}{2}$$

$$\langle \sin(kr-\omega t)\cos(kr-\omega t)\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin(kr-\omega t)\cos(kr-\omega t)\,dt \qquad \Big|\ \text{Subst}\quad \sin(kr-\omega t) = u$$

$$\frac{du}{dt} = -\omega\cos(kr-\omega t)$$

$$= \frac{\omega}{2\pi}\int_{\sin(kr)}^{\sin(kr-2\pi)} -\frac{u}{\omega}\,du$$

$$= \frac{1}{4\pi}\Big[\,u^2\,\Big]_{\sin(kr)}^{\sin(kr-2\pi)}$$

$$= 0$$

$$\Rightarrow \langle \vec{S}\rangle = \frac{A^2\sin^2\theta}{\mu_0\omega r^2}\left[\left(\frac{1}{kr^2}-k\right)\cdot\frac{1}{2} - \frac{1}{k^2r^2}\cdot\frac{1}{2}\right](-\vec{e}_r)$$

$$= \frac{A^2\sin^2\theta}{2\mu_0\omega r^2}\cdot\frac{1+k^3r^2-k}{k^2r^2}\,\vec{e}_r$$

✓

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a)

In 3b) wird gezeigt

$B_{0y} = \frac{k}{\omega}E_{0x}$ und $B_{0y} = \frac{\omega}{k}\frac{1}{c^2}E_{0x}$

$$\Rightarrow \frac{k}{\omega}E_{0x} = \frac{\omega}{k}\frac{1}{c^2}E_{0x}$$

$$k = \frac{\omega}{c}$$

✓

/ b)

Statik: $\vec{\nabla}\vec{E}_0 = \vec{\nabla}\vec{B}_0 = 0$

$$\vec{\nabla}\times\vec{E}_0 = \vec{\nabla}\times\vec{B}_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Maxwell im Ladungsfreien:

$$\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0} = 0 \quad \Rightarrow \quad \vec{\nabla}\vec{E}_0 = 0$$

$$\vec{\nabla}\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}\vec{B}_0 = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\begin{pmatrix} 0 - E_{0y}ik \\ E_{0x}ik - 0 \\ \frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}\end{pmatrix} e^{i(kz-\omega t)} = +i\omega\,\vec{B}_0\,e^{i(kz-\omega t)}$$

$$\Rightarrow \quad B_{0x} = -\frac{k}{\omega}E_{0y}$$

$$\Rightarrow \quad B_{0y} = \frac{k}{\omega}E_{0x}$$

$$\Rightarrow \quad \frac{\partial E_{0y}}{\partial x} = \frac{\partial E_{0x}}{\partial y}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$\begin{pmatrix} 0 - B_{0y}ik \\ B_{0x}ik - 0 \\ \frac{\partial B_{0y}}{\partial x} - \frac{\partial B_{0x}}{\partial y}\end{pmatrix} e^{i(kz-\omega t)} = -\frac{i\omega}{c^2}\vec{E}_0\,e^{i(kz-\omega t)}$$

$$\Rightarrow \quad B_{0y} = \frac{\omega}{k}\frac{1}{c^2}E_{0x}$$

$$\Rightarrow \quad B_{0x} = -\frac{\omega}{k}\frac{1}{c^2}E_{0y}$$

$$\Rightarrow \quad \frac{\partial B_{0y}}{\partial x} = \frac{\partial B_{0x}}{\partial y}$$

Es gilt $\frac{\partial B_{0y}}{\partial x} = \frac{\partial B_{0x}}{\partial y}$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \begin{pmatrix}0\\0\\ \frac{\partial B_{0y}}{\partial x} - \frac{\partial B_{0x}}{\partial y}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

genauso:

$$\vec{\nabla}\times\vec{E}_0 = \begin{pmatrix}0\\0\\ \frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Zusammen mit *1 und *2 ist also gezeigt das $\vec{E}_0$ und $\vec{B}_0$ sich wie in der Elektrostatik verhalten ✓

c)

$$\vec{E}_0 = E_{0r}\,\vec{e}_r + E_{0\varphi}\,\vec{e}_\varphi$$

$$E(r=a,\varphi) = E_a\,\vec{e}_r$$

$$E(r=b,\varphi) = E_b\,\vec{e}_r$$

$$\vec{\nabla}\vec{E}_0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r E_{0r}\right) + \frac{1}{r}\frac{\partial E_{0\varphi}}{\partial \varphi} \overset{!}{=} 0 \quad *1$$

$$\vec{\nabla}\times\vec{E}_0 = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r E_{0\varphi}\right) - \frac{\partial}{\partial \varphi}E_{0r}\right)\vec{e}_z \overset{!}{=} \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \quad \frac{E_{0\varphi}}{r} + \frac{\partial}{\partial r}E_{0\varphi} - \frac{\partial}{\partial \varphi}E_{0r} \overset{!}{=} 0 \quad *2$$

$\vec{E}_0$ soll aufgrund der Rotationssymmetrie nur von r abhängen:

$$\vec{E}_0 = \vec{E}_0(r)$$

aus $*1$ wird

$$\frac{E_{0r}(r)}{r} + \frac{\partial E_{0r}(r)}{\partial r} = 0$$

$$\int_{E_a}^{E_{0r}} \frac{dE_r'}{E_r'} = \int_a^r -\frac{dr'}{dr}$$

$$\left[\ln E_r'\right]_{E_a}^{E_{0r}} = -\left[\ln r'\right]_a^r$$

$$\frac{E_{0r}}{E_a} = \frac{a}{r}$$

$$E_{0r} = \frac{a}{r}E_a \quad ✓$$

Aus *2 wird:

$$\frac{E_{0\varphi}}{r} + \frac{\partial E_{0\varphi}}{\partial r} = 0$$

$$\int_0^{E_{0\varphi}} \frac{dE_\varphi'}{E_\varphi'} = -\int_a^r \frac{dr'}{r'}$$

$$\left[\ln E_\varphi'\right]_0^{E_{0\varphi}} = -\left[\ln r'\right]_a^r$$

$$\ln E_{0\varphi} - \ln 0 = -\ln r + \ln a$$

$$E_{0\varphi} = e^{\ln\left(\frac{0\cdot a}{r}\right)}$$

$$= 0$$

$$\Rightarrow \vec{E}_0 = \frac{a}{r} E_a \vec{e}_r \quad ✓$$

$$\Rightarrow \vec{E}_0 = \frac{a}{r} E_a \sin\varphi\, \vec{e}_y + \frac{a}{r} E_a \cos\varphi\, \vec{e}_x$$

Aus Aufgabe b): $B_{0x} = -\frac{k}{\omega} E_{0y}$

$B_{0y} = \frac{k}{\omega} E_{0x}$

$$\Rightarrow \vec{B}_0 = -\frac{a}{r}\frac{k}{\omega} E_a \sin\varphi\, \vec{e}_x + \frac{a}{r}\frac{k}{\omega} E_a \cos\varphi\, \vec{e}_y$$

$$= \frac{a}{r}\frac{k}{\omega} E_a \vec{e}_\varphi \quad ✓$$

d)

$$\oint \vec{E}\, d\vec{A} = \frac{1}{\varepsilon_0}\int \rho\, dV$$

$$E_a \cos(kz-\omega t)\, 2\pi a l = \frac{\rho}{\varepsilon_0} \pi a^2 l$$

$$\rho = 2\varepsilon_0 \frac{E_a}{a} \cos(kz-\omega t) \quad ✓$$

$$\oint B\, dl = \mu_0 I$$

$$2\pi a \frac{k}{\omega} E_a \cos(kz-\omega t) = \mu_0 I$$

$$I = 2\pi a \frac{1}{c\mu_0} \cos(kz-\omega t) \quad ✓$$