① 28/30 $\Delta \phi(\vec{r}) = 0$ $\quad \vec{r} = x e_x + y \hat{e}_y + z \hat{e}_z$

85/100
(Henry Seebach

$$\phi(x,y,z) = A(x)\, B(y)\, C(z) \quad (2)$$

a) 6/6 $\Rightarrow \Delta\phi = (\partial_x^2 A(x)) B(y) C(z) + (\partial_y^2 B(y)) A(x) C(z) + (\partial_z^2 C(z)) A(x) B(y) \;\Big|\; A(x)B(y)C(z)$

$$= \underbrace{\frac{\partial_x^2 A}{A}}_{:= f(x)} + \underbrace{\frac{\partial_y^2 B}{B}}_{:= g(y)} + \underbrace{\frac{\partial_z^2 C}{C}}_{:= h(z)} = 0 \quad \checkmark$$

$f(x) + g(y) + h(z) = 0 \Leftrightarrow$ Da die Funktionen von verschiedenen Variablen abhängen, muss jede Funktion konstant sein, damit die Gleichung erfüllt ist.

$$\Leftrightarrow \frac{\partial_x^2 A}{A} = C_1 \quad \frac{\partial_y^2 B}{B} = C_2 \quad \frac{\partial_z^2 C}{C} = C_3$$

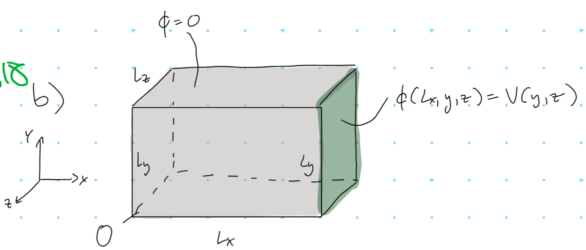
$$\Rightarrow \partial_x^2 A = C_1 A, \quad \partial_y^2 B = C_2 B, \quad \partial_z^2 C = C_3 C \quad \checkmark$$

mit $C_1 = \alpha^2$, $C_2 = \beta^2$, $C_3 = \gamma^2$

und $\alpha^2 + \beta^2 + \gamma^2 = 0$ folgt Gleichung (3): ✓

$$\partial_x^2 A = \alpha^2 A, \quad \partial_y^2 B = \beta^2 B, \quad \partial_z^2 C = \gamma^2 C \quad \text{mit } \alpha^2 + \beta^2 + \gamma^2 = 0$$

8/8 b)



z: $\phi(x,y,z) = \sum_{m,n=1}^{\infty} d_{m,n} \sinh\left(\sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}\, x\right) \sin\left(\frac{m\pi y}{L_y}\right) \sin\left(\frac{n\pi z}{L_z}\right) \quad (4)$

Randbedingungen:

$A(x=0) = B(y=0) = C(z=0) = 0 \Leftrightarrow a_- = -a_+,\; b_- = -b_+,\; c_- = -c_+$ ✓

- $B(0) = B(L_y) = 0 \Rightarrow$ Ansatz $B(y) = C_1 e^{\beta y} + C_2 e^{-\beta y}$

$B(0) = C_1 + C_2 = 0 \Leftrightarrow C_2 = -C_1$

$B(L_y) = C_1 \left(e^{\beta L_y} - e^{-\beta L_y} \right) = C_2 \underbrace{\left(e^{\beta L_y} - e^{-\beta L_y} \right)}_{\text{muss 0 sein}} = 0$

$\Rightarrow \beta = i\frac{m\pi}{L_y} \Rightarrow B(L_y) = \underbrace{e^{im\pi}}_{=-1} - \underbrace{e^{-im\pi}}_{=-1} = 0$ ✓

$B(y) = C_1 \left(e^{\frac{im\pi}{L_y}y} - e^{-\frac{im\pi}{L_y}y} \right) = C_1 \cdot 2i \sin\left(\frac{m\pi}{L_y} y\right) \overset{2i \text{ absorbiert}}{=} C_1 \sin\left(\frac{m\pi}{L_y} y\right)$ ✓

- Analog für C: $C(0) = C(L_z) = 0 \Rightarrow \gamma = \frac{n\pi}{L_z} \Rightarrow C(z) = C_1 \sin\left(\frac{n\pi}{L_z} z\right)$

- $A(0) = 0$, $A(L_x) = V(y,z) \Rightarrow$ Ansatz $A(x) = C_3 e^{i\alpha x} + C_4 e^{-i\alpha x}$

$A(0) = C_3 + C_4 \Rightarrow C_3 = -C_4 \Rightarrow C_4 \left(e^{i\alpha x} - e^{-i\alpha x} \right)$

$$\alpha^2 + \beta^2 + \gamma^2 = 0 \Leftrightarrow \alpha^2 = -(\beta^2 + \gamma^2) = i^2(\beta^2 + \gamma^2)$$

$$\alpha = i\sqrt{\beta^2 + \gamma^2}$$

$$A(x) = C_4 \left(e^{-\sqrt{\beta^2+\gamma^2}\,x} - e^{\sqrt{\beta^2+\gamma^2}\,x} \right) \overset{\sqrt{2} \text{ absorbiert}}{=} C_4 \Big(\underbrace{e^{\sqrt{\beta^2+\gamma^2}\,x} - e^{-\sqrt{\beta^2+\gamma^2}\,x}}_{= 2\sinh\left(\sqrt{\beta^2+\gamma^2}\,x\right)} \Big)$$ ✓

$$\Rightarrow \phi(x,y,z) = \sum_{m,n=1}^{\infty} d_{m,n} A(x) B(y) C(z)$$

$$\sum_{m,n=1}^{\infty} d_{m,n} \sinh\left(\sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}\, x \right) \sin\left(\frac{m\pi y}{L_y}\right) \sin\left(\frac{n\pi z}{L_z}\right)$$ ✓

Aus dem Skript Kap. 3.11.1:

$$d_{m,n} = \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2}\, L_x \right)} \int_0^{L_y}\int_0^{L_z} V(y,z) \sin\left(\frac{m\pi}{L_y} y\right) \sin\left(\frac{n\pi}{L_z} z\right) dy\, dz$$

6 | 6

c) $V(y,z) = \sin\left(\frac{\pi y}{L_y}\right)\sin\left(\frac{\pi z}{L_z}\right) = \phi(L_x, y, z)$. Man kann erkennen, dass von $\phi(L_x, y, z)$ nur das Summenglied mit $m = n = 1$ übrig bleibt.

$$= d_{11} \sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_z}\right)^2}\, L_x\right) \sin\left(\frac{\pi y}{L_y}\right) \sin\left(\frac{\pi z}{L_z}\right)$$

$$d_{11} = \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)} \int_0^{L_y}\int_0^{L_z} V(y,z) \sin\left(\frac{\pi}{L_y} y\right) \sin\left(\frac{\pi}{L_z} z\right) dy\, dz$$

$$= \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)} \int_0^{L_y}\int_0^{L_z} \sin^2\left(\frac{\pi}{L_y} y\right) \sin^2\left(\frac{\pi}{L_z} z\right) dy\, dz$$

Subst. $u = \frac{\pi y}{L_y}$, $t = \frac{\pi z}{L_z}$

$du = \frac{\pi}{L_y} dy$ $\quad dt = \frac{\pi}{L_z} dz$

$\sin^2(u) = \frac{1}{2} - \frac{1}{2}\cos(2u)$

$$= \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)} \int_0^{\pi} \frac{L_y}{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos(u)\right) du \int_0^{\pi} \frac{L_z}{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos(t)\right) dt$$

$$= \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)} \left[\frac{L_y}{2\pi}\left(u - \sin(u)\right)\right]_0^{\pi} \cdot \left[\frac{L_z}{2\pi}\left(t - \sin(t)\right)\right]_0^{\pi}$$

$$= \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)} \cdot \left(\frac{L_y}{2}\right) \cdot \left(\frac{L_z}{2}\right) = \frac{1}{\sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)}$$

$$\Rightarrow \phi(x,y,z) = \frac{\sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_z}\right)^2}\, x\right)}{\sinh\left(\sqrt{\left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_y}\right)^2}\, L_x\right)} \sin\left(\frac{\pi y}{L_y}\right) \sin\left(\frac{\pi z}{L_z}\right)$$

8/7G

d) $V(y,z) = y(y-L_y)\, z(z-L_z) \quad (y^2 - yL_y)(z^2 - zL_z) = \phi(L_x, y, z)$

$$= \sum_{m,n=1}^{\infty} d_{m,n} \sinh\left(\sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}\, L_x\right) \sin\left(\frac{m\pi y}{L_y}\right) \sin\left(\frac{n\pi z}{L_z}\right)$$

$$d_{m,n} = \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2}\, L_x\right)} \int_0^{L_y}\int_0^{L_z} V(y,z) \sin\left(\frac{m\pi}{L_y} y\right) \sin\left(\frac{n\pi}{L_z} z\right) dy\, dz$$

$$= \frac{4}{L_y L_z \sinh\left(\sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2}\, L_x\right)} \int_0^{L_y}\int_0^{L_z} (y^2 - yL_y)(z^2 - zL_z) \sin\left(\frac{m\pi}{L_y} y\right) \sin\left(\frac{n\pi}{L_z} z\right) dy\, dz$$

- $\int_0^{L_y} (y^2 - yL_y) \sin\left(\frac{m\pi y}{L_y}\right) dy$ | mit partieller Integration

$$= \underbrace{\left[(y^2 - yL_y)\left(-\frac{L_y}{m\pi} \cos\left(\frac{m\pi y}{L_y}\right)\right)\right]_0^{L_y}}_{=0} - \int_0^{L_y} \overset{f}{(2y - L_y)} \overset{g'}{\left(-\frac{L_y}{m\pi} \cos\left(\frac{m\pi y}{L_y}\right)\right)} dy$$

$$= -\left[\left[(2y - L_y)\left(-\frac{L_y^2}{m^2\pi^2} \sin\left(\frac{m\pi y}{L_y}\right)\right)\right]_0^{L_y} - \int_0^{L_y} 2\left(-\frac{L_y^2}{m^2\pi^2} \sin\left(\frac{m\pi y}{L_y}\right)\right) dy\right]$$

$$= \frac{L_y^3}{m^2\pi^2} \overbrace{\sin(m\cdot\pi)}^{=0} + \left[\frac{2L_y^3}{m^3\pi^3} \cos\left(\frac{m\pi y}{L_y}\right)\right]_0^{L_y}$$

$$= -\frac{2L_y^3}{m^3\pi^3} \overbrace{\cos(m\pi)}^{=(-1)^m} - \frac{2L_y^3}{m\pi^3}$$

$$= 2\left(\frac{L_y}{m\pi}\right)^3 \left(\underbrace{\cos(m\pi)}_{=-1,\ m\ \text{ungerade}} - 1\right) = \begin{cases} 0, & m \text{ gerade} \\ -4\left(\frac{L_y}{m\pi}\right)^3, & m \text{ ungerade} \end{cases}$$

✓

+ Gesamtergebnis

② 38/40

$$Y^*_{lm}(\theta,\varphi) = \sqrt{\frac{2l-1}{} \frac{(l-m)!}{(l+m)!}}\, e^{im\varphi} P_l^m(\cos\theta)$$

$$P_l^m(x) = \begin{cases} (-1)^m (1-x^2)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^m P_l(x), & m>0 \\ P_l(x) & m=0 \\ (-1)^{-m} \frac{(l+m)!}{(l-m)!} P_l^{-m}(x) & m<0 \end{cases}$$

a) 5/5 $Q = \int_{\mathbb{R}^3} \rho_0 \frac{R}{r^2} (R-2r)\, \theta(R-r) \sin\theta \, d^3r$

$$= \rho_0 \int_0^R \int_0^{2\pi} \int_0^{\pi} R\sin^2\theta - 2r\sin^2\theta \, d\theta dr = 2\pi \rho_0 \int_0^{\pi} (R^2 - R^2)\sin^2\theta \, d\theta = 0$$

$$q_{0,0} = \int_{\mathbb{R}^3} \rho(r)\, r^0\, Y^*_{00}(\theta,\varphi)\, d^3r = \frac{1}{\sqrt{4\pi}} Q = 0$$

b) 10/10 $l=1, \quad m=-1:$

$$q_{1,-1} = \int_{\mathbb{R}^3} d^3r\, \rho(\vec{r})\, r^1\, Y^*_{1,-1}(\theta,\varphi)$$

$$= \int_{\mathbb{R}^3} d^3r\, \rho(\vec{r})\, r^1 \sqrt{\frac{3}{4\pi} \frac{2!}{0!}}\, e^{-i\varphi} \cdot P_1^{-1}$$

$$P_1^{-1}(\cos\theta) = -\frac{1}{2} P_1^1(\cos\theta) = -\frac{1}{2}\left(-\sqrt{1-\cos^2\theta}\ \frac{d}{d\cos\theta}\cos\theta\right)$$

$$= \frac{1}{2}\sqrt{1-\cos^2\theta} = \frac{1}{2}\sin\theta$$

$$\Rightarrow q_{1,-1} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho_0 R r (R-2r) \sqrt{\frac{3}{8\pi}}\ e^{-i\varphi} \sin^3\theta \, dr\, d\theta\, d\varphi \quad ✓$$

$$= \rho_0 \sqrt{\frac{3}{8\pi}} \cdot \left[-i e^{-i\varphi}\right]_0^{2\pi} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^R \rho_0 R r (R-2r)\, dr$$

$$= \rho \ \frac{3}{8\pi} \Big[\underbrace{-i e^{-i2\pi} + i}_{=0}\Big]\left[-\cos\theta\right]_0^{\pi} \left[\frac{1}{2}R^2 r^2 - \frac{2}{3} R r^3\right]_0^R = 0 \quad ✓$$

$$\text{mit } e^{-i2\pi} = \underbrace{\cos(-2\pi)}_{=1} + i \underbrace{\sin(2\pi)}_{=0} = 1$$

$l = 1,\ m = 0$:

$$q_{1,0} = \int_{\mathbb{R}^3} d^3r\ \rho(\vec r)\, r\, Y^*_{1,0}(\theta,\varphi) = \int_{\mathbb{R}^3} d^3r\ \rho(\vec r)\, r \sqrt{\frac{3}{4\pi}\frac{1!}{1!}}\ \frac{1}{2}\frac{d}{d\cos\theta}(\cos^2\theta - 1)$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho_0 R r (R-2r) \sqrt{\frac{3}{4\pi}}\ \cos\theta \sin^2\theta \ dr\, d\theta\, d\varphi$$

$$\left|\ u = \sin^2\theta,\quad \frac{du}{d\theta} = 2\sin\theta\cos\theta,\quad d\theta = \frac{du}{2\sin\theta\cos\theta}\right.$$

$$= \int_0^{2\pi}\int_0^{R} \rho_0 R r (R-2r) \sqrt{\frac{3}{4\pi}}\ dr\, d\varphi \underbrace{\int_0^0 du\ \frac{\sin\theta}{2}\ d\theta}_{=0} = 0 \quad ✓$$

$l=1,\ m=1:$

$$q_{1,1} = \int_{\mathbb{R}^3} d^3r\ \rho(\vec{r})\, r\, Y^*_{1,1}(\theta,\varphi) = \int_{\mathbb{R}^3} d^3r\, \rho(\vec{r})\, r \sqrt{\frac{3}{8\pi}}\, e^{i\varphi}\, P_1^1(\cos\theta)$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} -\rho_0 R r(2-r) \sqrt{\frac{3}{8\pi}}\, e^{i\varphi} \sqrt{1-\cos^2\theta}\ \sin\theta\ dr\, d\theta\, d\varphi$$

$$= \underbrace{\Big[e^{i\varphi}\Big]_0^{2\pi}}_{=1-1=0} \int_0^{\pi}\int_0^{R} \rho(\vec{r})\, r \sqrt{\frac{3}{8\pi}}\, \sin^2\theta\, d\theta = 0 \quad ✓$$

5/5

c) $q_{l,-m} = (-1)^m q_{lm} \quad \left(Y_{l,-m} = (-1)^m Y^*_{l,m} \right) \implies q_{lm} = 0,\ m>0 \iff q_{lm} = 0,\ m<0$

$$q_{lm} = \int d^3r\ \rho(\vec{r})\, r^l\, Y^*_{lm}(\theta,\varphi) = \rho_0 \int_0^{R}\int_0^{2\pi}\int_0^{\pi} r^l R(R-2r) \sin^2\theta (-1)^m Y_{l,m}\ d\theta\, d\varphi\, dr$$

$$= \rho_0 \int_0^{R}\int_0^{2\pi}\int_0^{\pi} r^l R(R-2r) \sin^2\theta \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\varphi} (-1)^m (1-\cos^2\theta)^{\frac{m}{2}} \left(\frac{d}{d\cos\theta}\right)^m \frac{1}{2^l l!} \left(\frac{d}{d\cos\theta}\right)^l \underbrace{(\cos^2-1)^l}_{=\sin^2\theta}\ d\theta\, d\varphi\, dr$$

$$= \rho_0 \underbrace{\Big[e^{im\varphi}\Big]_0^{2\pi}}_{\cos(m2\pi)+i\sin(m2\pi)-(\cos(0)+i\sin(0)) = 1-1=0,\ m\neq 0} \int_0^{R}\int_0^{\pi} r^l R(R-2r) \sin^2\theta \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} (-1)^m (1-\cos^2\theta)^{\frac{m}{2}} \cdot \left(\frac{d}{d\cos\theta}\right)^{m+l} \frac{1}{2^l l!} \underbrace{(\cos^2-1)^l}_{=\sin^2\theta}\ d\theta\, d\varphi\, dr$$

✓

$$= 0$$

d) 10/10

$$q_{20} = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho_0 r^2 R(R-2r) \sin^2\theta \, Y_{20}(\theta,\varphi) \, d\theta \, d\varphi \, dr$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho_0 r^2 R(R-2r) \sin^2\theta \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right) d\theta \, d\varphi \, dr$$

$$= \rho_0 \sqrt{\frac{5}{16\pi}} \int_0^R (R^2 r^2 - 2Rr^3) \, dr \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^2\theta(3(1-\sin^2\theta)-1) \, d\theta \quad \Big| \; \cos^2\theta = 1-\sin^2\theta$$

$$= \rho_0 \sqrt{\frac{5}{16\pi}} \left[\frac{R^2}{3} r^3 - \frac{R}{2} R r^4\right]_0^R \cdot 2\pi \int_0^{\pi} (2\sin^2\theta - 3\sin^4\theta) \, d\theta$$

$$= \rho_0 \sqrt{\frac{5}{4}\pi} \, \frac{1}{6} R^5 \int_0^{\pi} (3\sin^4\theta - 2\sin^2\theta) \, d\theta \quad \Big| \; \sin^4\theta = \frac{1}{8}(3 - 4\cos(2\theta) + \cos(4\theta)), \; \sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$$

$$= \rho_0 \sqrt{\frac{5}{4}\pi} \, \frac{R^5}{6} \int_0^{\pi} \frac{3}{8}(3 - 4\cos(2\theta) + \cos(4\theta)) - (1 - \cos(2\theta)) \, d\theta$$

$$= \rho_0 \sqrt{\frac{5}{4}\pi} \, \frac{R^5}{6} \int_0^{\pi} \left(\frac{9}{8} - \frac{3}{4}\cos(2\theta) + \frac{3}{8}\cos(4\theta) - \frac{8}{8} + \frac{4}{4}\cos(2\theta)\right) d\theta$$

$$= \rho_0 \sqrt{\frac{5}{4}\pi} \, \frac{R^5}{6} \int_0^{\pi} \left(\frac{1}{8} - \frac{1}{4}\cos(2\theta) + \frac{3}{8}\cos(4\theta)\right) d\theta$$

$$= \rho_0 \sqrt{\frac{5}{4}\pi} \, \frac{R^5}{6} \left[\frac{1}{8}\theta - \frac{1}{8}\sin(2\theta) + \frac{3}{32}\sin(4\theta)\right]_0^{\pi}$$

$$= \rho_0 \sqrt{\frac{5}{4}\pi} \, \frac{R^5}{6} \cdot \frac{\pi}{8} = \rho_0 \sqrt{\frac{5}{4}\pi} \, \pi \, \frac{R^5}{48}$$ ✓

✓

e) 5/5

$$q_{l,0} = \int_{\mathbb{R}^3} \rho(\vec{r}) (r)^l P_l \cos(\theta) = \int_{\mathbb{R}^3} \rho_0 R(R-2r) \Theta(R-r) \sin^2\theta \, P_l(\cos\theta) \, r^l \, d\theta \, d\varphi \, dr$$

$$= \rho_0 \int_0^R R(R-2r) r^l \, d\theta \int_0^{2\pi} d\varphi \left(\int_0^{\frac{\pi}{2}} \sin^2\theta \, P_l(\cos\theta) \, d\theta + \int_{\frac{\pi}{2}}^{\pi} \sin^2\theta \, P_l(\cos\theta) \, d\theta \right.$$

subst. $u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = -\frac{du}{\sin\theta}$, $\sin\theta = \sin(\arccos(u))$

$$\Rightarrow \rho_0 \int_0^R R(R-2r) r^l d\theta \int_0^{2\pi} d\varphi \left(\int_1^0 -du \sin^2(\arccos(u)) P_l(u) + \int_0^{-1} -du \sin^2(\arccos(u)) P_l(u) \right)$$

subst. $z = -u \Rightarrow -du = dz$:

$$\int_0^{-1} -du \sin^2(\arccos(u)) P_l(u) = \int_0^1 dz \sin^2(\arccos(-z)) P_l(-z)$$

$$= \int_0^1 dz \sin^2(-\arccos(z))(-P_l(z))$$

$$= \int_0^1 dz \sin^2(\arccos(z)) P_l(z)$$

$$\Rightarrow \rho_0 \int_0^R R(R-2r) r^l d\theta \int_0^{2\pi} d\varphi \left(\underbrace{\int_0^1 du \sin^2(\arccos(u)) P_l(u) + \int_0^{-1} -du \sin^2(\arccos(u)) P_l(u)}_{=0} \right)$$

✓

f) 315 im Anhang!

③ 19/30

$$\phi(\vec r) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{1}{r^3}\sum_{i=1}^{3} p_i r_i + \frac{1}{2}\frac{1}{r^5}\sum_{i,j=1}^{3} Q_{ij} r_i r_j\right.$$

$Q^{(0)} = q,\ Q_i^{(1)} = p_i,\ Q_{ij}^{(2)} = Q_{ij}$

6/6 a) q bei $\vec r = \vec 0 \iff \rho(\vec r) = \delta(x)\delta(y)\delta(z)\, q$ ✓

$$Q^0 = \int \rho(\vec r)\, d^3r = q \quad ✓$$

$$\vec Q^{(1)}\hat e_x = \int x\, q\, \delta(x)\, d^3r = 0,\ \vec Q^{(1)}\hat e_y = \int y\, q\, \delta(y)\, d^3r = 0,\ \vec Q^{(1)}\hat e_z = \int z\, q\, \delta(z)\, d^3r = 0$$

$$Q_{ij}^{(2)} = \int d^3r\, (3x_i x_j - (x^2+y^2+z^2)\delta_{ij})\, q\, \delta(x,y,z)\Big|_{\vec r = \vec 0} = 0 \quad ✓$$

$$\Rightarrow \phi(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{(keine Näherung)} \quad ✓$$

b) 3/6 $\rho(\vec r) = \delta(x-a, y, z)\, q$ ✓

$$Q^{(0)} = \int \delta(x-a, y, z)\, q\, d^3r = q \quad ✓$$

$$\vec Q^{(1)}\hat e_x = \int x\, \delta(x-a, y, z)\, q\, d^3r = aq$$

$$\vec Q^{(1)}\hat e_y = \int y\, \delta(x-a, y, z)\, q\, d^3r = 0 \quad ✓$$

$$\vec Q^{(1)}\hat e_z = \int y\, \delta(x-a, y, z)\, q\, d^3r = 0$$

$$Q_{ij}^{(2)} = \int (x_i x_j - (x^2+y^2+z^2)\delta_{ij})\, q\, \delta(x-a, y, z)\, d^3r$$

$$= (x_i x_j - (x^2+y^2+z^2))\delta_{ij}\, q\Big|_{x=a,\ y=0,\ z=0} = q(x^2 - x^2) = 0 \quad f$$

für $i=j$, sonst auch 0

$$\Rightarrow \phi(\vec r) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{a q x}{r^3}\right) + \text{Quadrupol}$$

6/6 c) $\rho(\vec r) = \delta(x+\frac{a}{2}, y, z)\, q - \delta(x-\frac{a}{2}, y, z)\, q$

$$Q^{(0)} = \int \delta(x+\tfrac{a}{2}, y, z)\, q - \delta(x-\tfrac{a}{2}, y, z)\, q \; d^3r = q - q = 0$$

$$\vec Q^{(1)} \hat e_x = \int (\delta(x+\tfrac{a}{2}, y, z)\, q - \delta(x-\tfrac{a}{2}, y, z)\, q)\, x \; d^3r = -\tfrac{a}{2}q - \tfrac{a}{2}q$$
$$= -aq$$

$$\vec Q^{(1)} \hat e_y = \int (\delta(x+\tfrac{a}{2}, y, z)\, q - \delta(x-\tfrac{a}{2}, y, z)\, q)\, y = 0$$

$$\vec Q^{(1)} \hat e_z = \int (\delta(x+\tfrac{a}{2}, y, z)\, q - \delta(x-\tfrac{a}{2}, y, z)\, q)\, y = 0$$

$$Q^{(2)}_{ij} = \int (x_i x_j - (x^2+y^2+z^2)\delta_{ij})(\delta(x+\tfrac{a}{2}, y, z)\, q - \delta(x-\tfrac{a}{2}, y, z)\, q)\, d^3r$$

$$= (x_i x_j - (x^2+y^2+z^2)\delta_{ij}\, q \Big|_{x=-\frac{a}{2}, y=0, z=0}$$

$$- (x_i x_j - (x^2+y^2+z^2)\delta_{ij}\, q \Big|_{x=\frac{a}{2}, y=0, z=0} = 0$$

$$\Rightarrow \phi(\vec r) \approx -\frac{1}{4\pi\varepsilon_0} \frac{a q x}{r^3}$$

d) 4/6 $\rho(\vec r) = 2q\,\delta(x+\frac{a}{2}, y, z) - q\,\delta(x-\frac{a}{2}, y, z)$

$$Q^{(0)} = \int 2q\,\delta(x+\tfrac{a}{2}, y, z) - q\,\delta(x-\tfrac{a}{2}, y, z)\; d^3r = q$$

$$\vec Q^{(1)} \hat e_x = \int (2q\,\delta(x+\tfrac{a}{2}, y, z) - q\,\delta(x-\tfrac{a}{2}, y, z))\, x \; d^3r$$

$$= -q \cdot a - q\,\tfrac{a}{2} = -\tfrac{3}{2} a q$$

$$\vec{Q}^{(1)}\hat{e}_y = \int \left(2q\,\delta\left(x+\frac{a}{2}, y, z\right) - q\,\delta\left(x-\frac{a}{2}, y, z\right)\right) z \, d^3r = 0$$

$$\vec{Q}^{(1)}\hat{e}_z = \int \left(2q\,\delta\left(x+\frac{a}{2}, y, z\right) - q\,\delta\left(x-\frac{a}{2}, y, z\right)\right) z \, d^3r = 0 \quad ✓$$

$$Q^{(2)}_{ij} = \int \left(x_i x_j - (x^2+y^2+z^2)\,\delta_{ij}\right)\left(2q\,\delta\left(x+\frac{a}{2}, y, z\right) - q\,\delta\left(x-\frac{a}{2}, y, z\right)\right) d^3r$$

$$= 2q\left(x_i x_j - (x^2+y^2+z^2)\delta_{ij}\right)\Big|_{x=-\frac{a}{2},\, y=z=0} - q\left(x_i x_j - (x^2+y^2+z^2)\delta_{ij}\right)\Big|_{x=\frac{a}{2},\, y=z=0} = 0 \; f$$

$$\Rightarrow \phi(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{3}{2}aq\frac{x}{r^3}\right) \; ✓ \; + \text{Quadrupol}$$

f e), f)

Theo C Blatt Nr.5

Aufgabe 2 f)

Das Gauß'sche Gesetz ist für solche Ladungsverteilungen eher ungeeignet, weil durch die nicht kugelsymmetrische Ladungsverteilung das Integral: $\int_V \vec{E}\,\vec{n}\,d^3 r$ schwer lösbar wird.

Für den Fall $\sum_i q_i = 0$ funktioniert das Gauß'sche Gesetz ebenfalls nicht, da $\int_V \rho(\vec{r})\,d^3 r = 0$,obwohl gelten darf: $\rho(\vec{r}) \neq 0$.

Berechnung des Multipols q_{40}:

In[8]:=
```
Q04 = ρ0 * Integrate[r^4 * R * (R - 2 * r) * Sin[θ]^2 * SphericalHarmonicY[4, 0, θ, ϕ],
    {r, 0, R}, {ϕ, 0, 2 * Pi}, {θ, 0, Pi}]
```

Out[8]= $\frac{1}{320}\pi^{3/2} R^7 \rho_0$ ✓

Plot von: Φ_{20}, Φ_{40}, $\epsilon = \frac{\Phi_{40}}{\Phi_{20}}$

In[9]:=
```
ϵ0 := QuantityMagnitude[Entity["PhysicalConstant", "ElectricConstant"]["Value"]]
ρ0 := 1
R := 1

SphericalHarmonicY[2, 0, θ, ϕ]
SphericalHarmonicY[4, 0, θ, ϕ]

q20 = ρ0 * Integrate[r^2 * R * (R - 2 * r) * Sin[θ]^2 * SphericalHarmonicY[2, 0, θ, ϕ],
    {r, 0, R}, {ϕ, 0, 2 * Pi}, {θ, 0, Pi}]
q40 = ρ0 * Integrate[r^4 * R * (R - 2 * r) * Sin[θ]^2 * SphericalHarmonicY[4, 0, θ, ϕ],
    {r, 0, R}, {ϕ, 0, 2 * Pi}, {θ, 0, Pi}]
```

Out[12]=

$$\frac{1}{4}\sqrt{\frac{5}{\pi}}\left(-1 + 3\,\text{Cos}[\theta]^2\right)$$

Out[13]=

$$\frac{3\left(3 - 30\,\text{Cos}[\theta]^2 + 35\,\text{Cos}[\theta]^4\right)}{16\sqrt{\pi}}$$

Out[14]=

$$\frac{1}{96}\sqrt{5}\,\pi^{3/2}$$

Out[15]=

$$\frac{\pi^{3/2}}{320}$$

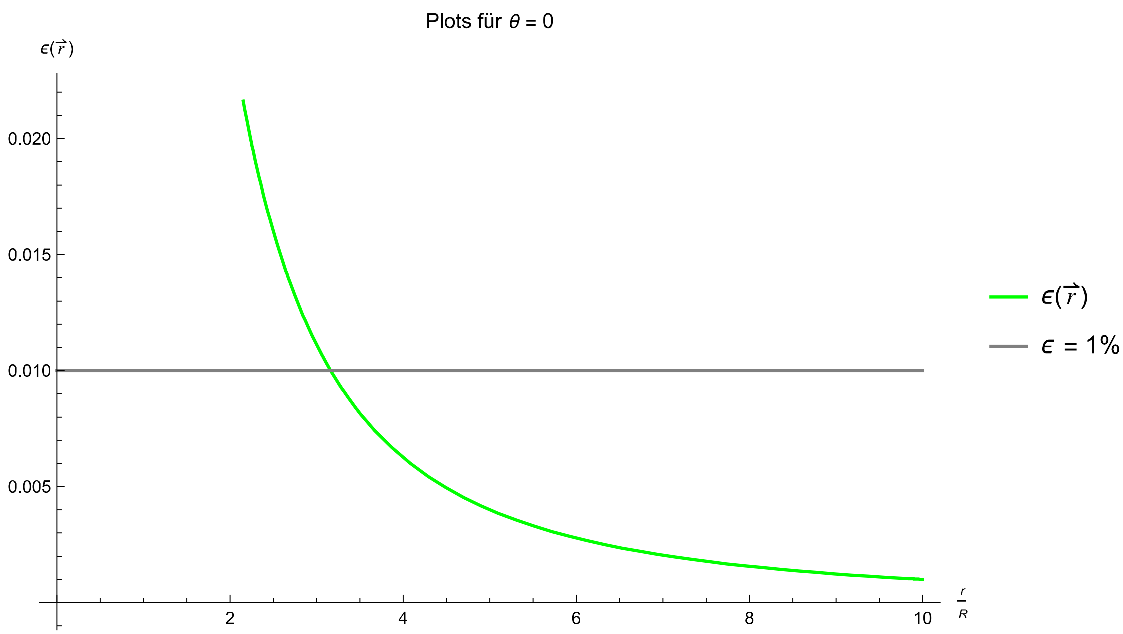
In[16]:=

```
phi20[r_, θ_] :=
  (1 / ϵ0) * (1 / 5) * (1 / r^3) * (1/96 * √5 * π^(3/2) * (R^5)) * (1/4 * Sqrt[5/π] (-1 + 3 Cos[θ]^2))
phi40[r_, θ_] :=
  (1 / ϵ0) * (1 / 9) * (1 / r^5) * (1/320 π^(3/2) R^7) * ((3 (3 - 30 Cos[θ]^2 + 35 Cos[θ]^4)) / (16 √π))


phi20[d_, θ_] :=
  (1 / ϵ0) * (1 / 5) * (1 / d^3) * (1/96 * √5 * π^(3/2) * (R^2)) * (1/4 * Sqrt[5/π] (-1 + 3 Cos[θ]^2))
phi40[d_, θ_] :=
  (1 / ϵ0) * (1 / 9) * (1 / d^5) * (1/320 π^(3/2) R^2) * ((3 (3 - 30 Cos[θ]^2 + 35 Cos[θ]^4)) / (16 √π))
ϵ[d_, θ_] := Abs[phi40[d, θ] / phi20[d, θ]];

(*Fall: θ=0*)
Plot[{phi20[d, 0], phi40[d, 0]}, {d, R, 5 * R}, PlotRange → {-10^8, 10^8},
  PlotLegends → {"ϕ20", "ϕ40"}, AxesLabel → {"r/R", "ϕ(r⃗)"},
  PlotLabel → "Plots für θ = 0", ImageSize → 500, PlotStyle → {Blue, Red, Thick}]
Plot[{Abs[phi40[d, 0] / phi20[d, 0]], 0.01}, {d, 0, 10}, ImageSize → 500,
  AxesLabel → {"r/R", "ϵ(r⃗)"}, PlotLabel → "Plots für θ = 0",
  PlotStyle → {Green, Gray}, PlotLegends → {"ϵ(r⃗)", "ϵ = 1%"}]

(*Fall: θ=π/2*)
Plot[{phi20[d, Pi / 2], phi40[d, Pi / 2]}, {d, R, 5 * R}, PlotRange → {-10^8, 10^8},
  PlotLegends → {"ϕ20", "ϕ40"}, AxesLabel → {"r/R", "ϕ(r⃗)"},
  PlotLabel → "Plots für θ = π/2", ImageSize → 500, PlotStyle → {Blue, Red, Thick}]
Plot[{Abs[phi40[d, Pi / 2] / phi20[d, Pi / 2]], 0.01}, {d, 0, 10},
  ImageSize → 500, AxesLabel → {"r/R", "ϵ(r⃗)"}, PlotStyle → {Green, Gray},
  PlotLabel → "Plots für θ = π/2", PlotLegends → {"ϵ(r⃗)", "ϵ = 1%"}]
```

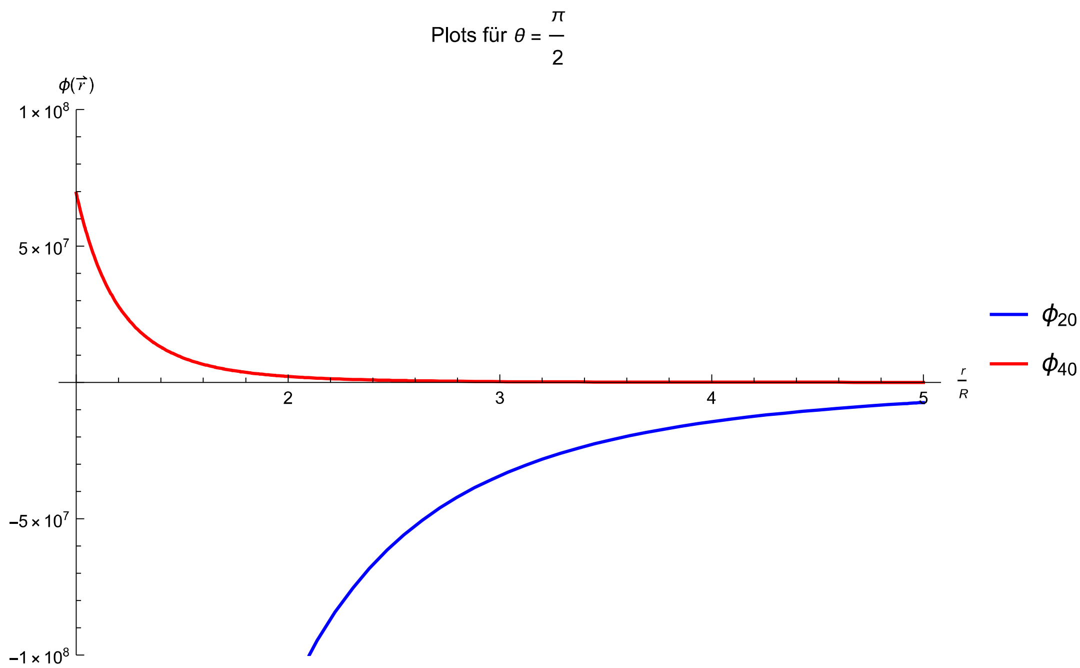
Out[21]=


Plots für θ = 0
ϕ(r⃗)
1 × 10^8
5 × 10^7
−5 × 10^7
−1 × 10^8
2
3
4
5
r/R
ϕ20
ϕ40
Besser Log Log Plot

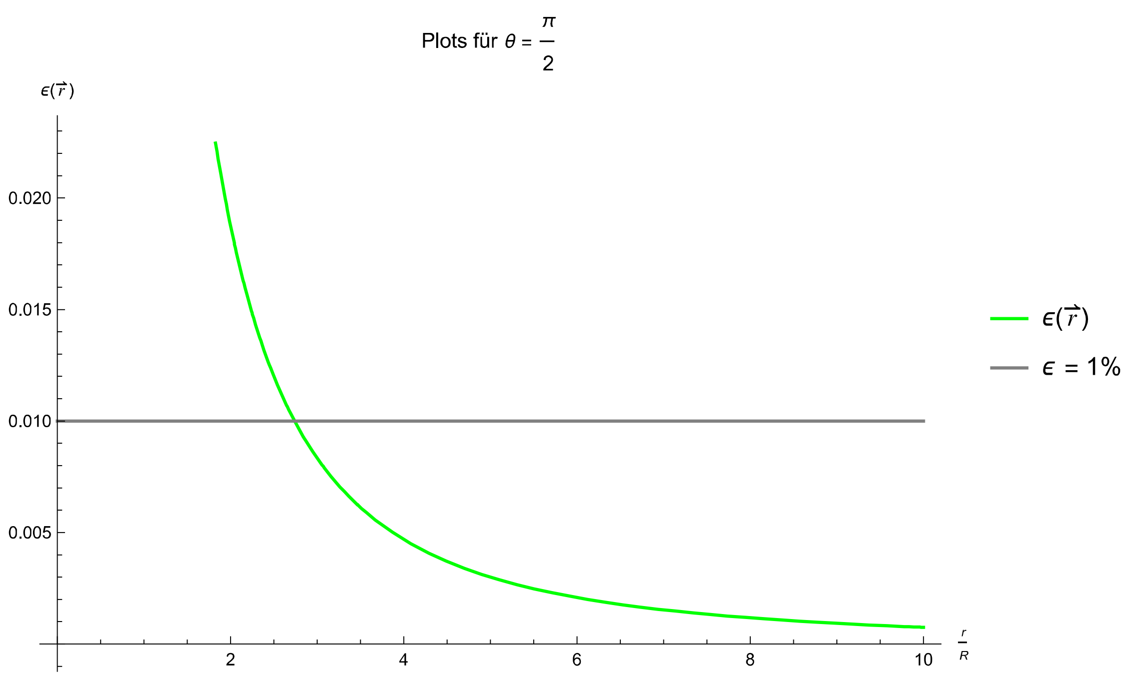
Out[22]=


Plots für θ = 0
ϵ(r⃗)
0.020
0.015
0.010
0.005
2
4
6
8
10
r/R
ϵ(r⃗)
ϵ = 1%

Out[23]=



Out[24]=



Mit $\epsilon(r_1, \Theta) = \dfrac{\phi_{40}}{\phi_{20}} = 0.01$ folgt die Kurve: $r_1(\Theta) = \dfrac{R^2}{\sqrt{7}} \sqrt{\left| \dfrac{35\, cos^4(\Theta) - 30\, cos^2(\Theta) + 3}{0.01} \right|}$

F

In[26]:=
```
r1[θ_] := R^2 / (5)^0.5 * (Abs[(35 * Cos[θ]^4 - 30 * Cos[θ]^2 + 3) / 0.01])^0.5
Plot[r1[θ], {θ, 0, Pi},
 PlotLabel → "Abstand r mit ϵ(r,θ) = 0.01 in Abhängigkeit von θ",
 AxesLabel → {"θ in Bogenmaß", "r"}, PlotStyle → {Yellow, Brown}, ImageSize → 500]
```

Out[27]=