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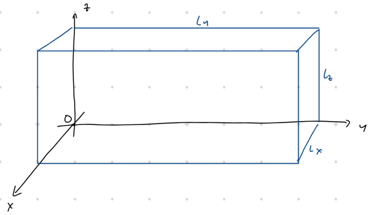

$\hat{n} \times \vec{E}(\vec{r},t) = 0$, $\hat{n}\vec{B}(\vec{r},t) = 0$

a) $\Box \vec{E} = 0 \iff (\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\vec{E} = 0$

b) $E_x(\vec{r},t) = X(x)\,Y(y)\,Z(z)\,T(t)$

$\hookrightarrow (\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2)E_x = 0$

$\iff X''(x)\,Y(y)\,Z(z)\,T(t) + X(x)\,Y''(y)\,Z(z) + X(x)Y(y)Z''(z) - X(x)Y(y)\,Z(z)\,T''(t) = 0$

$$\iff \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} - \frac{1}{c^2}\frac{T''(t)}{T(t)} = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -k_x^2 , \quad \frac{Y''(y)}{Y(y)} = -k_y^2 , \quad \frac{Z''(z)}{Z(z)} = -k_z^2 , \quad \frac{T''(t)}{T(t)} = -\omega^2$$

$$\iff -(k_x^2 + k_y^2 + k_z^2) + \frac{\omega^2}{c^2} = 0$$

$$\iff \omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)$$

c) $X'' = -k_x^2 X \Rightarrow X = x_0 \sin(k_x x + \varphi_x)$

$Y'' = -k_y^2 Y \Rightarrow Y = y_0 \sin(k_y y + \varphi_y)$

$Z'' = -k_z^2 Z \Rightarrow Z = z_0 \sin(k_z z + \varphi_z)$

$T'' = -\omega^2 T \Rightarrow T = t_2 e^{-i\omega t}$

$$E_x(\vec{r},t) = \mathrm{Re}\left[c_x \sin(k_x x + \varphi_x)\sin(k_y y + \varphi_y)\sin(k_z z + \varphi_z)\, e^{-i\omega t} \right]$$

$\hat{n} \times \vec{E}(\vec{r},t) = 0$: $\underline{\hat{n} = \hat{e}_y}$: $\hat{e}_y \times \vec{E}\big|_{y=0} = \hat{e}_y \times (E_x\hat{e}_x + E_y\hat{e}_y + E_z\hat{e}_z)\big|_{y=0} = -\hat{e}_z E_x(y=0) + \hat{e}_x \overbrace{E_z(y=0)}^{=0} = 0$ $\Rightarrow E_x(y=0) = 0$

: $\hat{e}_y \times \vec{E}\big|_{y=L_y} = -\hat{e}_z E_x(y=L_y) + \hat{e}_x \underbrace{E_z(y=L_y)}_{=0} = 0 \Rightarrow E_x(y=L_y) = 0$

$\underline{\hat{n} = \hat{e}_z}$: $\hat{e}_z \times \vec{E}\big|_{z=0} = \hat{e}_z \times (E_x\hat{e}_x + E_y\hat{e}_y + E_z\hat{e}_z)\big|_{z=0} = \hat{e}_y E_x(z=0) - \hat{e}_x \underbrace{E_y(z=0)}_{=0} = 0$

$\Rightarrow E_x(z=0) = 0$

$\hat{e}_z \times \vec{E}\big|_{z=L_z} = \hat{e}_y E_x(z=L_z) - \hat{e}_x \underbrace{E_y(z=L_z)}_{=0} = 0$

$\Rightarrow E_x(z=L_z) = 0$

$\hookrightarrow E_x(y=0) = Re\left[C_x \sin(k_x x + \varphi_x)\sin(\varphi_y)\sin(k_z z + \varphi_z)\, e^{-i\omega t}\right] = 0 \quad \Rightarrow \varphi_y = 0$

$\hookrightarrow E_x(y=L_y) = Re\left[C_x \sin(k_x x + \varphi_x)\sin(k_y L_y)\sin(k_z z + \varphi_z)\, e^{-i\omega t}\right] \Rightarrow k_y = \frac{m\pi}{L_y}$

$\hookrightarrow E_x(z=0) = Re\left[C_x \sin(k_x x + \varphi_x)\sin(\frac{m\pi y}{L_y})\sin(\varphi_z)\, e^{-i\omega t}\right] = 0 \quad \Rightarrow \varphi_z = 0$

$\hookrightarrow E_x(z=L_z) = Re\left[C_x \sin(k_x x + \varphi_x)\sin(\frac{m\pi y}{L_y})\sin(k_z L_z)\quad e^{-i\omega t}\right] = 0 \Rightarrow L_z = \frac{n\pi}{L_z}$

$\Rightarrow E_x(\vec{r},t) = Re\left[C_x \sin(k_x x + \varphi_x)\sin(\frac{m\pi y}{L_y})\sin(\frac{n\pi z}{L_z})\, e^{-i\omega t}\right]$

d) $$\vec{\nabla}\vec{E} = 0 = \vec{\nabla}\Big[Re\left[C_x \sin(k_x x + \varphi_x)\sin(\frac{m\pi y}{L_y})\sin(\frac{n\pi z}{L_z})\, e^{-i\omega t}\right]\hat{e}_x$$
$$+ Re\left[C_y \sin(\frac{l'\pi x}{L_x})\sin(k_y y + \alpha_y')\sin(\frac{n'\pi z}{L_z})\, e^{-i\omega t}\right]\hat{e}_y$$
$$+ Re\left[C_z \sin(\frac{l''\pi x}{L_x})\sin(\frac{m''\pi y}{L_y})\sin(k_z z + \alpha_z'')\, e^{-i\omega'' t}\right]\hat{e}_z\Big]$$
$$= Re\Big[C_x k_x \cos(k_x x + \varphi_x)\sin(\frac{m\pi y}{L_y})\sin(\frac{n\pi z}{L_z})\, e^{-i\omega t}$$
$$+ C_y \sin(\frac{l'\pi x}{L_x})\, k_y \cos(k_y y + \alpha_y')\sin(\frac{n'\pi z}{L_z})\, e^{-i\omega t}$$
$$+ C_z \sin(\frac{l''\pi x}{L_x})\sin(\frac{m''\pi y}{L_y})\, k_z \cos(k_z z + \alpha_z'')\, e^{-i\omega'' t}\Big] = 0 \ \forall x,y,z,t$$

Wahr für $l = l' = l''$, $m = m' = m''$, $n = n' = n''$, $\omega = \omega' = \omega''$, $\varphi_x = -\frac{\pi}{2}$

$\Rightarrow 0 = k_x C_x + k_y C_y + k_z C_z \qquad k_x = \frac{l\pi}{L_x}$

e) $\vec{\nabla}\times\vec{E} = -\partial_t \vec{B} \Rightarrow \vec{B} = -\int dt\, \vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\int dt\, \vec{E} = -\vec{\nabla}\times\frac{1}{-i\omega}\vec{E} = -\frac{i}{\omega}\vec{\nabla}\times\vec{E}$

f) $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) = c^2\left((\frac{L\pi}{L_x})^2 + (\frac{m\pi}{L_y})^2 + (\frac{n\pi}{L_z})^2\right)$

$\Rightarrow \omega_{lmn} = c\sqrt{(\frac{L\pi}{L_x})^2 + (\frac{m\pi}{L_y})^2 + (\frac{n\pi}{L_z})^2}$

g) $\omega_{lmn} = c\cancel{\pi}\sqrt{\left(\frac{l}{L}\right)^2 + \left(\frac{m}{L}\right)^2 + \left(\frac{n}{L}\right)^2} < \frac{4\pi \cancel{c}}{L}$

$$\sqrt{\left(\frac{l}{L}\right)^2 + \left(\frac{m}{L}\right)^2 + \left(\frac{n}{L}\right)^2} < \frac{4}{L}$$

$$l^2 + m^2 + n^2 < 16$$

(2) $\vec{E}(\vec{r},t) = X(x)\,Y(y)\,Z(z)\,T(t)$

- $\left(\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right) E_i \hat{e}_i = 0$

$\Leftrightarrow \hat{e}_i \; X_i'' Y_i Z_i T_i + X_i Y_i'' Z_i T_i + X_i Y_i Z_i'' T_i - \frac{1}{c^2} T_i'' X_i Y_i Z_i = 0$

$\Leftrightarrow \frac{X_i''}{X_i} + \frac{Y_i''}{Y_i} + \frac{Z_i''}{Z_i} - \frac{1}{c^2}\frac{T_i''}{T_i} = 0$

$\Leftrightarrow \alpha_i + \beta_i + \gamma_i - \frac{1}{c^2}\sigma_i = 0$

$\Leftrightarrow -k_{x,i}^2 - k_{y,i}^2 - k_{z,i}^2 - \frac{1}{c^2}\omega_i^2 = 0$

$\Rightarrow X_i(x) = X_{0,i} \sin(k_{x,i} + \alpha_{x,i})$

$Y_i(y) = Y_{0,i} \sin(k_{y,i} + \alpha_{y,i})$

$Z_i(z) = Z_{0,i}\, e^{-ik_{z,i} z} + \tilde{Z}_{0,i}\, e^{ik_{z,i} z}$

$T_i(t) = T_{0,i}\, e^{-i\omega_i t} + \tilde{T}_{0,i}\, e^{i\omega_i t}$

$\Rightarrow \vec{E}(\vec{r},t) = \sum_{i=x,y,z} \hat{e}_i \; c_i \sin(k_{x,i}\, x + \alpha_{x,i}) \sin(k_{y,i} + \alpha_{y,i})\, e^{-i\omega t}$