(1) a) $\rho(\vec{r},t) = q(\delta(\vec{r}-d\hat{e}_z) - \delta(\vec{r}+d\hat{e}_z))\, e^{-i\omega t}$

$\vec{j}(\vec{r},t) = -i\omega\vec{p}\,\delta(\vec{r})e^{-i\omega t}$ mit $\vec{p} = p\hat{e}_z = 2qd\,\hat{e}_z$ für $d \ll r$

$$\vec{A}(\vec{r},t) = \int_{\mathbb{R}^3} d^3r' \frac{\mu_0}{4\pi} \frac{\vec{j}(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$ ✓

$$= \int_{\mathbb{R}^3} d^3r' \frac{\mu_0}{4\pi} \frac{-i\omega \overbrace{2qd\,\hat{e}_z}^{=\vec{p}}\,\delta(\vec{r}')\, e^{-i\omega(t - \frac{|\vec{r}-\vec{r}'|}{c})}}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{\mu_0}{2\pi} \frac{i\omega qd\, e^{-i\omega t + i\omega\frac{r}{c}}}{r}\hat{e}_z$$

$$= -\frac{\mu_0\, i\omega qd\, e^{ikr}}{2\pi r}\hat{e}_z\, e^{-i\omega t} = -\frac{\mu_0 i\omega p\, e^{-i\omega(t-\frac{r}{c})}}{4\pi r}\hat{e}_z$$ ✓

b)
$$\phi(\vec{r},t) = \int d\vec{r}' \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$ ✓

$$\approx \int d\vec{r}' -\frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\vec{\nabla}\delta(\vec{r}')e^{-i\omega t}}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{1}{4\pi\varepsilon_0}\int d\vec{r}'\; 2qd\hat{e}_z \overbrace{\vec{\nabla}\delta(\vec{r}')}^{f'} \overbrace{\frac{e^{-i\omega(t-\frac{|\vec{r}-\vec{r}'|}{c})}}{|\vec{r}-\vec{r}'|}}^{g}$$ ✓ | partielle Integration

durch Auswerten an den Grenzen Null

$$= -\frac{qd}{2\pi\varepsilon_0}\hat{e}_z\left[\left[\delta(\vec{r}')\frac{e^{-i\omega(t-\frac{|\vec{r}-\vec{r}'|}{c})}}{|\vec{r}-\vec{r}'|}\right] - \int d\vec{r}'\; \delta(\vec{r}')\; \vec{\nabla}\frac{e^{-i\omega t}\, e^{\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right]$$ ✓

$$= \frac{qd}{2\pi\varepsilon_0}\hat{e}_z \int d\vec{r}'\; e^{-i\omega t}\delta(\vec{r}')\left(-\frac{i\omega}{c}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}e^{\frac{i\omega}{c}|\vec{r}-\vec{r}'|} + \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}e^{\frac{i\omega}{c}|\vec{r}-\vec{r}'|}\right)$$

$$= \frac{qd}{2\pi\varepsilon_0}\hat{e}_z\, e^{-i\omega t}\left(-\frac{i\omega}{c}\frac{\vec{r}}{r^2}e^{\frac{i\omega r}{c}} + \frac{\vec{r}}{r^3}e^{\frac{i\omega r}{c}}\right)$$

$$= \frac{\vec{p}}{4\pi\varepsilon_0}\left(\frac{r\hat{e}_r}{r^3} - \frac{i\omega r\hat{e}_r}{cr^2}\right)e^{-i\omega(t-\frac{r}{c})}$$

$$= \frac{\vec{p}}{4\pi\varepsilon_0}\hat{e}_r\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)e^{-i\omega(t-\frac{r}{c})}$$ ✓

c) $\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t)$ ✓

$$\left(\hat{e}_r \partial_r + \hat{e}_\theta \frac{1}{r}\partial_\theta + \hat{e}_\varphi \frac{1}{r\sin\theta}\partial_\varphi\right) \times \hat{e}_z \left(-\frac{\mu_0}{4\pi} \frac{i\omega p\, e^{-i\omega(t-\frac{r}{c})}}{r}\right)$$

keine θ- oder φ-Abhängigkeit

$$= (\hat{e}_r \times \hat{e}_z)\, \partial_r \frac{-\mu_0 i\omega p\, e^{-i\omega(t-\frac{r}{c})}}{4\pi r}$$

$$= (\hat{e}_r \times \vec{p}) \left(-\frac{\mu_0 i^2 \omega^2}{4\pi r c} + \frac{\mu_0 i\omega}{4\pi r^2}\right) e^{-i\omega(t-\frac{r}{c})}$$

$$= \frac{\omega^2 \mu_0}{4\pi c} (\hat{e}_r \times \vec{p}) \left(\frac{1}{r} - \frac{c}{i\omega r^2}\right) e^{-i\omega(t-\frac{r}{c})}$$

$\mu_0 = \frac{1}{c^2 \varepsilon_0}$

$$= \frac{\omega^2}{4\pi c^3 \varepsilon_0} (\hat{e}_r \times \vec{p}) \left(\frac{1}{r} - \frac{c}{i\omega r^2}\right) e^{-i\omega(t-\frac{r}{c})} \quad //$$

✓

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \partial_t \vec{A}(\vec{r},t)$$ ✓

$$= -\frac{\partial\phi}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi + \partial_t\left(\frac{\mu_0 i\omega p\, e^{-i\omega(t-\frac{r}{c})}}{4\pi r}\hat{e}_z\right)$$

NR: $$\partial_r\phi = \partial_r\left(\frac{\vec{p}\hat{e}_r}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)e^{-i\omega(t-\frac{r}{c})}\right) = \frac{\vec{p}\hat{e}_r}{4\pi\varepsilon_0}\left[-\frac{2}{r^3} + \frac{i\omega}{r^2 c} + \frac{i\omega}{cr^2} - \frac{i^2\omega^2}{c^2 r}\right]e^{-i\omega(t-\frac{r}{c})}$$

$$= \frac{\vec{p}\hat{e}_r}{4\pi\varepsilon_0}\left(-\frac{2}{r^3} + \frac{2i\omega}{r^2 c} + \frac{\omega^2}{c^2 r}\right)e^{-i\omega(t-\frac{r}{c})}$$

$$\partial_\theta\phi = \partial_\theta\left(\frac{p\cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)e^{-i\omega(t-\frac{r}{c})}\right) = -\frac{p\sin\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)e^{-i\omega(t-\frac{r}{c})}$$

$$\partial_\varphi\phi = 0 \quad , \quad \vec{p}\cdot\hat{e}_r = p\cos\theta$$

$$= \frac{\omega^2 p}{4\pi r c^2\varepsilon_0}$$

$$= \left[-\frac{p\cos\theta}{4\pi\varepsilon_0}\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}\left(-\frac{2}{r^3} + \frac{2i\omega}{r^2 c} + \frac{\omega^2}{c^2 r}\right) + \frac{p\sin\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^3} - \frac{i\omega}{cr^2}\right)\hat{e}_\theta + \overbrace{\frac{\mu_0\omega^2 p}{4\pi r}}\hat{e}_z\right]e^{-i\omega(t-\frac{r}{c})}$$

$$= \left[\frac{\omega^2}{4\pi\varepsilon_0 r c^2}\left(-p\cos\theta\,\hat{e}_r + \hat{p}_{e_z}\right) + \frac{p\sin\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^3} - \frac{i\omega}{cr^2}\right)\hat{e}_\theta - \frac{p\cos\theta}{4\pi\varepsilon_0}\left(-\frac{2}{r^3} + \frac{2i\omega}{r^2 c}\right)\hat{e}_r\right]e^{-i\omega(t-\frac{r}{c})}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{\omega^2}{c^2 r}\begin{pmatrix}-p\cos\theta\sin\theta\cos\varphi\\ -p\cos\theta\sin\theta\sin\varphi\\ p - p\cos^2\theta\end{pmatrix} + \left(\frac{1}{r^3} - \frac{i\omega}{cr^2}\right)\begin{pmatrix}p\sin\theta\cos\theta\cos\varphi\\ p\sin\theta\cos\theta\sin\varphi\\ -p\sin^2\theta\end{pmatrix}\right.$$

$$\left. + \left(\frac{1}{r^3} - \frac{i\omega}{r^2 c}\right)\begin{pmatrix}2p\cos\theta\sin\theta\cos\varphi\\ 2p\cos\theta\sin\theta\sin\varphi\\ 2p\cos^2\theta\end{pmatrix}\right]e^{-i\omega(t-\frac{r}{c})}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{\omega^2}{c^2 r}\begin{pmatrix}-p\cos\theta\sin\theta\cos\varphi\\ -p\cos\theta\sin\theta\sin\varphi\\ p - p\cos^2\theta\end{pmatrix} + \left(\frac{1}{r^3} - \frac{i\omega}{cr^2}\right)\begin{pmatrix}3p\cos\theta\sin\theta\cos\varphi\\ 3p\cos\theta\sin\theta\sin\varphi\\ \underbrace{2p\cos^2\theta - p\sin^2\theta}\end{pmatrix}\right]e^{-i\omega(t-\frac{r}{c})}$$ ✓

$$= 2p\cos^2\theta - p(1-\cos^2\theta)$$

$$= 3p\cos^2\theta - p$$ //

ja passt. Dennoch umständlich

Mit Auflösen der "anderen Seite" erhält man das auf dem Blatt angegebene Ergebnis:

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{\omega^2}{c^2 r}(\hat{e}_r\times\vec{p})\times\hat{e}_r+\left(\frac{1}{r^3}-\frac{i\omega}{cr^2}\right)\left(3\hat{e}_r(\hat{e}_r\cdot\vec{p})-\vec{p}\right)\right]e^{-i\omega(t-\frac{r}{c})}$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{\omega^2}{c^2 r}\begin{pmatrix}-p\sin\theta\cos\varphi\cos\theta\\-p\sin\theta\sin\varphi\cos\theta\\p\sin^2\theta\end{pmatrix}+\left(\frac{1}{r^3}-\frac{i\omega}{cr^2}\right)\left(3p\begin{pmatrix}\cos\theta\sin\theta\cos\varphi\\\cos\theta\sin\theta\sin\varphi\\\cos^2\theta\end{pmatrix}-\vec{p}\right)\right]e^{-i\omega(t-\frac{r}{c})}$$

$p\sin^2\theta = p(1-\cos^2\theta) = p-p\cos^2\theta$

✓

NR: $(\hat{e}_r\times\vec{p})\times\hat{e}_r=\begin{pmatrix}p\sin\theta\sin\varphi\\-p\sin\theta\cos\varphi\\0\end{pmatrix}\times\begin{pmatrix}\sin\theta\cos\varphi\\\sin\theta\sin\varphi\\\cos\theta\end{pmatrix}=\begin{pmatrix}-p\sin\theta\cos\varphi\cos\theta\\-p\sin\theta\sin\varphi\cos\theta\\p\sin^2\theta\sin^2\varphi+p\sin^2\theta\cos^2\varphi\end{pmatrix}$

$$=\begin{pmatrix}-p\sin\theta\cos\varphi\cos\theta\\-p\sin\theta\sin\varphi\cos\theta\\p\sin^2\theta\end{pmatrix}$$

$\vec{S}=\frac{1}{\mu_0}\left[\mathrm{Re}(\vec{E})\times\mathrm{Re}(\vec{B})\right]$ ⚠

d) $\vec{S}=\frac{1}{\mu_0}(\vec{E}\times\vec{B})=\frac{1}{\mu_0}\left[\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega^2}{c^2 r}(\hat{e}_r\times\vec{p})\times\hat{e}_r+\left(\frac{1}{r^3}-\frac{i\omega}{cr^2}\right)\left(3\hat{e}_r(\hat{e}_r\cdot\vec{p})-\vec{p}\right)\right)e^{-i\omega(t-\frac{r}{c})}\right.$

$$\left.\times\frac{\omega^2}{4\pi\varepsilon_0 c^3}(\hat{e}_r\times\vec{p})\left(\frac{1}{r}-\frac{c}{i\omega r^2}\right)e^{-i\omega(t-\frac{r}{c})}\right]$$

−5P

$$=\frac{1}{\mu_0}\left[\frac{1}{4\pi\varepsilon_0}\frac{\omega^2}{c^2 r}\frac{\omega^2}{4\pi\varepsilon_0 c^3}\left(\frac{1}{r}-\frac{c}{i\omega r^2}\right)p^2\sin^2\theta\,\hat{e}_r\right.$$

$$\left.+\frac{1}{4\pi\varepsilon_0}\frac{\omega^2}{4\pi\varepsilon_0 c^3}\left(\frac{1}{r^3}-\frac{i\omega}{cr^2}\right)\left(\frac{1}{r}-\frac{c}{i\omega r^2}\right)p^2\hat{e}_r(3\cos^2\theta-1)\right]e^{-2i\omega(t-\frac{r}{c})}$$

$$=\frac{\omega^4}{\mu_0\varepsilon_0^2 16\pi^2 c^5 r}\left(\frac{1}{r}-\frac{c}{i\omega r^2}\right)p^2\sin^2\theta\,\hat{e}_r$$

$$+\frac{\omega^2}{16\pi^2\varepsilon_0^2\mu_0 c^3}\left(\frac{1}{r^3}-\frac{i\omega}{cr^2}\right)\left(\frac{1}{r}-\frac{c}{i\omega r^2}\right)p^2\hat{e}_r(3\cos^2\theta-1)\Big]e^{-2i\omega(t-\frac{r}{c})}$$

$$=\left[\frac{\omega^4}{16\pi^2\varepsilon_0 c^3}\left(\frac{1}{r^2}-\frac{c}{i\omega r^3}\right)p^2\sin^2\theta\,\hat{e}_r+\frac{\omega^2}{16\pi^2\varepsilon_0 c}\left(\frac{2}{r^4}-\frac{c}{i\omega r^5}-\frac{i\omega}{cr^3}\right)\right]e^{-2i\omega(t-\frac{r}{c})}$$

NR:

- $((\hat{e}_r \times \vec{p}) \times \hat{e}_r) \times (\hat{e}_r \times \vec{p}) = ((\hat{e}_r \cdot \hat{e}_r)\vec{p} - (\vec{p}\cdot\hat{e}_r)\hat{e}_r) \times (\hat{e}_r \times \vec{p})$

$= (\vec{p} - p\cos\theta\,\hat{e}_r) \times (\hat{e}_r \times \vec{p}) = \vec{p} \times (\hat{e}_r \times \vec{p}) - p\cos\theta\,\hat{e}_r \times (\hat{e}_r \times \vec{p})$

$= (\vec{p}\cdot\vec{p})\hat{e}_r - (\vec{p}\,\hat{e}_r)\vec{p} - p\cos\theta\,((\hat{e}_r\cdot\vec{p})\,\hat{e}_r - (\hat{e}_r\,\hat{e}_r)\,\vec{p})$

$= p^2\hat{e}_r - p\cos\theta\,\vec{p} - p\cos\theta\,(p\cos\theta\,\hat{e}_r - \vec{p})$

$= p^2\hat{e}_r - p^2\cos^2\theta\,\hat{e}_r = p^2\hat{e}_r(1-\cos^2\theta) = p^2\sin^2\theta\,\hat{e}_r$

- $(3\hat{e}_r(\hat{e}_r\cdot\vec{p}) - \vec{p}) \times (\hat{e}_r \times \vec{p}) = 3p\cos\theta\,\hat{e}_r \times (\hat{e}_r \times \vec{p}) - \vec{p} \times (\hat{e}_r \times \vec{p})$

$= 3p\cos\theta\,(\hat{e}_r\vec{p})\hat{e}_r - 3p\cos\theta\,(\hat{e}_r\cdot\hat{e}_r)\vec{p} - (\vec{p}\cdot\vec{p})\hat{e}_r + (\vec{p}\,\hat{e}_r)\vec{p}$

$= 3p^2\cos^2\theta\,\hat{e}_r - 3p\cos\theta\,\vec{p} - p^2\hat{e}_r + p\cos\theta\,\vec{p}$

$= p^2\hat{e}_r(3\cos^2\theta - 1)$

$\langle\vec{S}\rangle = \frac{1}{4\mu_0}(\vec{E}_0 \times \vec{B}_0^* + \vec{E}_0^* \times \vec{B}_0) = \frac{\mu_0}{32\pi^2}\frac{\omega^4}{r^2 c}(\hat{e}_r\vec{p}^2 - \hat{e}_r(\hat{e}_r\vec{p})^2)$ ✓

$|\hat{e}_r \times \vec{p}|\hat{e}_r$

e) $P = \int d\Omega\,\langle\vec{S}\rangle\hat{e}_r r^2 = \int_0^\pi\int_0^{2\pi} \frac{\mu_0}{32\pi^2}\frac{\omega^4}{r^2 c}(\hat{e}_r p^2 - \hat{e}_r(\hat{e}_r\vec{p})^2)\,\hat{e}_r r^2\sin\theta$

$= \frac{\mu_0\omega^4}{32\pi^2 c}\int_0^\pi\int_0^{2\pi}\underbrace{(p^2 - p^2\cos^2\theta)}_{=p^2(1-\cos^2\theta) = p^2\sin^2\theta}\sin\theta\,d\varphi\,d\theta = \frac{\mu_0\omega^4 p^2}{16\pi c}\underbrace{\int_0^\pi \sin^3\theta\,d\theta}_{=\frac{4}{3}} = \frac{\mu_0\omega^4 p^2}{12\pi c}$ ✓ //

f) $\vec{g}(\frac{\omega}{c}\hat{e}_r) = \int dr \; \vec{j}_0(\vec{r}')e^{-\frac{i\omega}{c}\hat{e}_r\vec{r}'} = -\int dr' \, i\omega\vec{p}\,\delta(\vec{r}')e^{-\frac{i\omega}{c}\hat{e}_r\vec{r}'} = -i\omega\vec{p}$ ✓

$\vec{B}_0(\vec{r}) = \frac{\mu_0 i\omega}{4\pi c}\frac{e^{i\frac{\omega}{c}r}}{r}\hat{e}_r \times \vec{g}(\frac{\omega}{c}\hat{e}_r) = \frac{\mu_0\omega^2 e^{i\frac{\omega}{c}r}}{4\pi cr}\hat{e}_r\times\vec{p}$ ✓

$\vec{E}_0(\vec{r}) = -\frac{\mu_0 i\omega}{4\pi}\frac{e^{i\frac{\omega}{c}r}}{r}\hat{e}_r\times(\hat{e}_r\times\vec{g}(\frac{\omega}{c}\hat{e}_r)) = \frac{\mu_0\omega^2 e^{i\frac{\omega}{c}r}}{4\pi r}(\hat{e}_r\times\vec{p})\times\hat{e}_r$ ✓

$\vec{B}(\vec{r},t) = \frac{\mu_0\omega^2}{4\pi rc}e^{-i\omega(t-\frac{r}{c})}\hat{e}_r\times\vec{p}$

$\vec{E}(\vec{r},t) = \frac{\mu_0\omega^2}{4\pi r^2}e^{-i\omega(t-\frac{r}{c})}(\hat{e}_r\times\vec{p})\times\hat{e}_r$

hier lässt sich nur $\langle\vec{S}\rangle$ finden

−2P

~~$\vec{S}$~~ $= \frac{1}{4\mu_0}(\vec{E}\times\vec{B} + \vec{E}^*\times\vec{B} + \vec{E}\times\vec{B}^* + \vec{E}^*\times\vec{B}^*)$

$= \frac{1}{\mu_0 c}(\hat{e}_r\vec{p}^2 - \hat{e}_r(\hat{e}_r\vec{p})^2)\left(\frac{\mu_0^2}{16\pi^2}\frac{\omega^4}{r^2}e^{-2i\omega(t-\frac{r}{c})} + \frac{\mu_0^2\omega^2}{8\pi^2r^2}e^{2i\omega(t-\frac{r}{c})}\right)$

$= \frac{\mu_0\omega^4}{32\pi^2cr^2}(\hat{e}_r\vec{p}^2 - \hat{e}_r(\hat{e}_r\vec{p})^2)(1+\cos(2\omega(t-\frac{r}{c})))$

$\langle\vec{S}\rangle = \frac{\mu_0\omega^4}{32\pi^2cr^2}(\hat{e}_r\vec{p}^2 - \hat{e}_r(\hat{e}_r\vec{p})^2) = \frac{\mu_0\omega^4}{32\pi^2cr^2}\sin^2\theta\, p^2\hat{e}_r$ ✓

$P = \int d\Omega\,\langle\vec{S}\rangle r^2\hat{e}_r = \int_0^{\pi}\int_0^{2\pi}\frac{\mu_0\omega^4}{32\pi^2c}\sin^3\theta\, p^2\, d\varphi\, d\theta = \frac{\mu_0}{12\pi c}\omega^4p^2$

g) im Anhang

48/55

② a) $\vec{j}(\vec{r},t) = I_0\cos(\omega t)\delta(\rho-R)\delta(z)e_\varphi$

$\vec{g}(k\hat{e}_r) = \int d\vec{r}'\,\vec{j}_0(\vec{r}')e^{-ik\hat{e}_r\vec{r}'} = \int d\vec{r}'\,\vec{j}_0(\vec{r}')(1-ik\hat{e}_r\vec{r}')$ mit $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\mathcal{O}(x^2)$

$= \int_0^{2\pi}\int_0^{\infty}\int_0^{\infty} I_0\rho'\,\delta(\rho'-R)\delta(z')\hat{e}_{\varphi'}(1-ikR\hat{e}_r\vec{r}')\,dz'\,d\rho'\,d\varphi'$

$= \int_0^{2\pi}d\varphi'\, I_0R\hat{e}_\varphi(1-ikR\sin\theta\cos\varphi) = -\int_0^{2\pi}d\varphi\; ik\, I_0R^2\hat{e}_{\varphi'}\sin\theta\cos\varphi'$ ✓

$= -ikR^2I_0\hat{e}_y\sin\theta\int_0^{2\pi}d\varphi'\cos^2\varphi' = -ikR^2I_0\hat{e}_y\sin\theta\,\pi$ ✓

$$\vec{B}_0 = \frac{\mu_0 ik}{4\pi} \frac{e^{ikr}}{r} \hat{e}_r \times \hat{e}_\varphi \left(-ikR^2 I_0 \pi \sin\theta\right) = -\frac{\mu_0 k^2 R^2 I_0}{4} \frac{e^{ikr}}{r} \sin\theta \, \hat{e}_\theta$$ ✓

$$\vec{E}_0 = -c\, \hat{e}_r \times \vec{B}_0 = \frac{\mu_0 k^2 R^2 I_0 c}{4} \frac{e^{ikr}}{r} \sin\theta \hat{e}_\varphi$$ ✓

b) $$\vec{S} = \frac{1}{4\mu_0}\left(\vec{E} \times \vec{B} + \vec{E}^* \times \vec{B} + \vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}^*\right)$$

$$= \frac{1}{4\mu_0} \frac{\mu_0^2 k^4 R^4 I_0^2 c}{16 r^2} \sin^2\theta \underbrace{\left(2 + e^{2i\omega(t-kr)} + e^{-2i\omega(t-\frac{k}{r})}\right)}_{= 2(1+\cos(2\omega(t-kr)))} \hat{e}_r$$

$$= \frac{\mu_0 \omega^4}{32 c^3} \frac{R^2 I_0^2}{r^2} \sin^2\theta \, \hat{e}_r \left[1 + \underbrace{\cos(2\omega(t - \tfrac{k}{r}))}\right]$$

wirkend die Phase nicht −1P

c) $$\langle \vec{S} \rangle = \frac{\mu_0 \omega^4}{32 c^3} \frac{R^2 I_0^2}{r^2} \sin^2\theta \, \hat{e}_r$$ ✓

$$P = \int d\Omega \, \langle \vec{S} \rangle r^2 \hat{e}_r = \int_0^{2\pi}\int_0^{\pi} \frac{\mu_0 \omega^4}{32 c^3} R^2 I_0^2 \sin^3\theta = 2\pi \frac{4}{3} \frac{\mu_0 \omega^4}{32 c^3} R^2 I_0^2$$

$$= \frac{\mu_0 \omega^4}{12 C^3} R^2 I_0^2$$ ✓

4 9/10

③ a) Entwicklung um $r' = 0$: $\rho(\vec{r},t) = \sum_{n=1}^{4} q_n \delta(\vec{r}-\vec{r}\,')e^{-i\omega t}$

$$= \sum_{n=1}^{4} q_n e^{-i\omega t}\left(\delta(\vec{r}) - (\vec{r}\,'\vec{\nabla})\delta(\vec{r}) + \frac{1}{2}(\vec{r}\,'\vec{\nabla})^2\delta(\vec{r}) + \mathcal{O}(|\vec{r}\,|^3)\right)$$

$\mathcal{O}(|\vec{r}\,|^3) = \mathcal{O}(a^3)$ vernachlässigbar wegen $a \ll r$ ✓

$\partial_t \rho + \vec{\nabla}\vec{j} = 0$ $\quad \vec{\nabla}\vec{j} = -\partial_t \rho = i\omega \sum_{n=1}^{4} q_n e^{-i\omega t}\left(\delta(\vec{r}) - \underbrace{(\vec{r}\,\vec{\nabla})\delta(\vec{r})}_{=\vec{r}\,'\nabla\delta(\vec{r})} + \frac{1}{2}\underbrace{(\vec{r}\,'\vec{\nabla})^2\delta(\vec{r})}_{=r_i' \partial_i r_j' \partial_j \delta(\vec{r})} + \mathcal{O}(|\vec{r}\,|^3)\right)$

$$\Rightarrow \vec{j} = i\omega e^{-i\omega t}\sum_{n=1}^{4} q_n\left(\frac{1}{4\pi}\frac{\hat{e}_r}{r^2} + \vec{r}_n\delta(\vec{r}) + \frac{1}{2}\vec{r}_n(\vec{r}_n\vec{\nabla})\delta(\vec{r})\right)$$

wie kommst du darauf − 1P

b) $q_1 = -q_3 = q$, $q_2 = q_4 = 0$ ✓

$$\vec{j} = i\omega e^{-i\omega t}\sum_{n=1}^{4} q_n\left(\frac{1}{4\pi}\frac{\hat{e}_r}{r^2} + \vec{r}_n\delta(\vec{r}) + \frac{1}{2}\vec{r}_n(\vec{r}_n\vec{\nabla})\delta(\vec{r})\right)$$

$$= i\omega e^{-i\omega t}\left[\frac{\hat{e}_r}{4\pi r^2}(q_1+q_3) + \delta(\vec{r})(a\hat{e}_x q_1 - a\hat{e}_x q_3) + \frac{1}{2}(a\hat{e}_x q_1 (a\hat{e}_x q_1 \vec{\nabla}) - a\hat{e}_x q_3(-a\hat{e}_x\vec{\nabla}))\right]$$

$$= i\omega e^{-i\omega t}\left[2qa\hat{e}_x\delta(\vec{r}) + \frac{1}{2}(aq\hat{e}_x(aq\,e_x\vec{\nabla}) - aq\hat{e}_x(a\hat{e}_x\vec{\nabla}))\right]$$ ✓

mit $\vec{g}(k\hat{r}) = \int d\vec{r}\,'\, \vec{j}_0(\vec{r}\,')e^{-ik\hat{r}\vec{r}\,'} = \int d\vec{r}\,'\, 2qa\,i\omega e^{-ik\hat{r}\vec{r}\,'}\hat{e}_x\delta(\vec{r}) = 2qai\omega\hat{e}_x$ folgt

die Strahlungscharakteristik $\frac{dP}{d\Omega} = \frac{c\mu_0 k^2}{32\pi^2}|\hat{e}_r \times \vec{g}|^2 = q\,a\,i\omega\frac{c\mu_0 k^2}{16\pi^2}\left|\begin{matrix}0\\ \cos\theta \\ -\sin\theta\sin\varphi\end{matrix}\right|^2$

$$= \frac{qai\omega c\mu_0 k^2}{16\pi^2}(\cos^2\theta + \sin^2\theta\sin^2\varphi)$$ ✓

f) Der Fall $q_n = q$ ist unphysikalisch, da der Term $i\omega e^{-i\omega t} \sum_{i=1}^{4} q_n \frac{1}{4\pi} \frac{\hat{e}_r}{r^2}$ nicht verschwindet, was der Kontinuitätsgleichung ✓ und Ladungserhaltung widerspricht:

$$Q_{ges} = \int dr \sum_{n=1}^{4} \delta(\vec{r})\, q_n e^{-i\omega t} = \sum_{n=1}^{4} q_n e^{-i\omega t} \neq \text{const. für } \sum_{n=1}^{4} q_n \neq 0$$ ✓

(c.) (d.), (e.) fehlt −24P

// 10/35

$\Sigma_p = 10 + 9 + 48$

$= 57/100$ LF

Theo C Blatt 10

Aufgabe 1 g)

Es werden die Realteile der berechneten Felder zum Zeitpunkt t=0 in der x,z-Ebene geplottet. Dabei werden alle Konstanten auf 1 gesetzt.

In[178]:=

```
ep0 := 1
k := 1
ω := 1
t := 1
c := 1
q := 1
d := 1
mu0 := 1

(* Felder aus Aufgabenteil c)/d) *)
BFeld[r_, θ_, ϕ_] := ω^2 / (4 * ep0 * Pi * c^2) *
  Cross[{0, 1, 0}, 2 * (Cos[θ] * {1, 0, 0} - Sin[θ])] * 1 / r * Cos[r * k - ω * t]
EFeld[r_, θ_, ϕ_] :=
 {1, 0, 0} * Cos[θ] * 2 / r^3 + Sin[θ] * {0, 1, 0} * (1 / r^3 - ω^2 / c^2 * 1 / r)
PoynVec[r_, θ_, ϕ_] := ω^2 * mu0 / (8 * Pi^2) * Cos[2 * k * r - 2 * ω * t] * 1 / r *
  (ω^2 / (c^2 * r) * Sin[θ]^2 * {1, 0, 0} - 1 / r^3 * Sin[θ]^2 * {1, 0, 0})

(* Felder aus Aufgabenteil f) *)
BFeldF[r_, θ_, ϕ_] := -1 / (4 * Pi) * 1 / r * Cos[k * r - ω * t] *
  Sin[θ] * Cross[{1, 0, 0}, (Cos[θ] * {1, 0, 0} - Sin[θ] * {0, 1, 0})]
EFeldF[r_, θ_, ϕ_] := -2 / (4 * Pi * r) * Cos[k * r - ω * t] * Sin[θ] * {0, 1, 0}
PoynVecF[r_, θ_, ϕ_] :=
 2 * 1 / (8 * Pi^2 * r^2) * Cos[k * r - ω * t]^2 * Sin[θ]^2 * {1, 0, 0}

plot1 =
 TransformedField["Spherical" → "Cartesian", BFeld[r, θ, ϕ], {r, θ, ϕ} → {x, y, z}]
plot2 =
 TransformedField["Spherical" → "Cartesian", EFeld[r, θ, ϕ], {r, θ, ϕ} → {x, y, z}]
plot3 = TransformedField[
  "Spherical" → "Cartesian", PoynVec[r, θ, ϕ], {r, θ, ϕ} → {x, y, z}]
plot4 = TransformedField[
  "Spherical" → "Cartesian", BFeldF[r, θ, ϕ], {r, θ, ϕ} → {x, y, z}]
plot5 = TransformedField[
  "Spherical" → "Cartesian", EFeldF[r, θ, ϕ], {r, θ, ϕ} → {x, y, z}]
plot6 = TransformedField["Spherical" → "Cartesian",
  PoynVecF[r, θ, ϕ], {r, θ, ϕ} → {x, y, z}]
```

Out[192]=

$$\left\{-\frac{x\sqrt{x^2+y^2}\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{2\pi\left(x^2+y^2+z^2\right)^{3/2}}-\frac{y\left(\frac{2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}-\frac{2z}{\sqrt{x^2+y^2+z^2}}\right)\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{4\pi\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}},\right.$$
$$-\frac{y\sqrt{x^2+y^2}\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{2\pi\left(x^2+y^2+z^2\right)^{3/2}}+\frac{x\left(\frac{2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}-\frac{2z}{\sqrt{x^2+y^2+z^2}}\right)\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{4\pi\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}},$$
$$\left.-\frac{\sqrt{x^2+y^2}\,z\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{2\pi\left(x^2+y^2+z^2\right)^{3/2}}\right\}$$

Out[193]=

$$\left\{\frac{2xz}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{xz\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{1}{\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2},\right.$$
$$\frac{2yz}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{yz\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{1}{\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2},$$
$$\left.\frac{2z^2}{\left(x^2+y^2+z^2\right)^{5/2}}-\frac{\left(x^2+y^2\right)\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}-\frac{1}{\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2}\right\}$$

Out[194]=

$$\left\{\frac{x\left(-\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}\right)\mathrm{Cos}\left[2-2\sqrt{x^2+y^2+z^2}\right]}{8\pi^2\left(x^2+y^2+z^2\right)},\right.$$
$$\frac{y\left(-\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}\right)\mathrm{Cos}\left[2-2\sqrt{x^2+y^2+z^2}\right]}{8\pi^2\left(x^2+y^2+z^2\right)},$$
$$\left.\frac{z\left(-\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}\right)\mathrm{Cos}\left[2-2\sqrt{x^2+y^2+z^2}\right]}{8\pi^2\left(x^2+y^2+z^2\right)}\right\}$$

Out[195]=

$$\left\{-\frac{y\sqrt{x^2+y^2}\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},\frac{x\sqrt{x^2+y^2}\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},0\right\}$$

Out[196]=

$$\left\{-\frac{xz\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{2\pi\left(x^2+y^2+z^2\right)^{3/2}},-\frac{yz\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{2\pi\left(x^2+y^2+z^2\right)^{3/2}},\frac{\left(x^2+y^2\right)\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\right]}{2\pi\left(x^2+y^2+z^2\right)^{3/2}}\right\}$$

Out[197]=

$$\left\{\frac{x\,(x^2+y^2)\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\,\right]^2}{4\,\pi^2\,(x^2+y^2+z^2)^{5/2}},\right.$$

$$\left.\frac{y\,(x^2+y^2)\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\,\right]^2}{4\,\pi^2\,(x^2+y^2+z^2)^{5/2}},\ \frac{(x^2+y^2)\,z\,\mathrm{Cos}\left[1-\sqrt{x^2+y^2+z^2}\,\right]^2}{4\,\pi^2\,(x^2+y^2+z^2)^{5/2}}\right\}$$

$$\text{VectorPlot}\left[\left\{-\frac{x\sqrt{x^2}\,\text{Cos}\left[1-\sqrt{x^2+z^2}\right]}{2\pi\left(x^2+z^2\right)^{3/2}},-\frac{\sqrt{x^2}\,z\,\text{Cos}\left[1-\sqrt{x^2+z^2}\right]}{2\pi\left(x^2+z^2\right)^{3/2}}\right\},\right.$$

```
 {x, -100, 100}, {z, -100, 100}, FrameLabel → {"x", "z"}, PlotRange → All,
 VectorColorFunction → "BlueGreenYellow",
 PlotLabel → "Magnetisches Feld aus Teilaufgabe c)", ImageSize → 500]
```

$$\text{VectorPlot}\left[\left\{\frac{2\,x\,z}{\left(x^2+z^2\right)^{5/2}}+\frac{x\,z\left(\frac{1}{\left(x^2+z^2\right)^{3/2}}-\frac{1}{\sqrt{x^2+z^2}}\right)}{x^2+z^2},\frac{2\,z^2}{\left(x^2+z^2\right)^{5/2}}-\frac{\left(x^2\right)\left(\frac{1}{\left(x^2+z^2\right)^{3/2}}-\frac{1}{\sqrt{x^2+z^2}}\right)}{x^2+z^2}\right\},\right.$$

```
 {x, -100, 100}, {z, -100, 100}, FrameLabel → {"x", "z"}, PlotRange → All,
 VectorColorFunction → "BlueGreenYellow",
 PlotLabel → "Elektrisches Feld aus Teilaufgabe c)", ImageSize → 500]
```

$$\text{VectorPlot}\left[\left\{\frac{x\left(-\frac{x^2}{\left(x^2+z^2\right)^{5/2}}+\frac{x^2}{\left(x^2+z^2\right)^{3/2}}\right)\text{Cos}\left[2-2\sqrt{x^2+z^2}\right]}{8\pi^2\left(x^2+z^2\right)},\frac{z\left(-\frac{x^2}{\left(x^2+z^2\right)^{5/2}}+\frac{x^2}{\left(x^2+z^2\right)^{3/2}}\right)\text{Cos}\left[2-2\sqrt{x^2+z^2}\right]}{8\pi^2\left(x^2+z^2\right)}\right\},\{x,-100,100\},\right.$$

```
 {z, -100, 100}, FrameLabel → {"x", "z"}, PlotRange → All,
 VectorColorFunction → "BlueGreenYellow",
 PlotLabel → "Poyting Vektor aus Teilaufgabe d)", ImageSize → 500]
```

```
VectorPlot[{0, 0}, {x, -100, 100},
 {z, -100, 100}, FrameLabel → {"x", "z"}, PlotRange → All,
 VectorColorFunction → "BlueGreenYellow",
 PlotLabel → "Magnetisches Feld aus Teilaufgabe f)", ImageSize → 500]
```

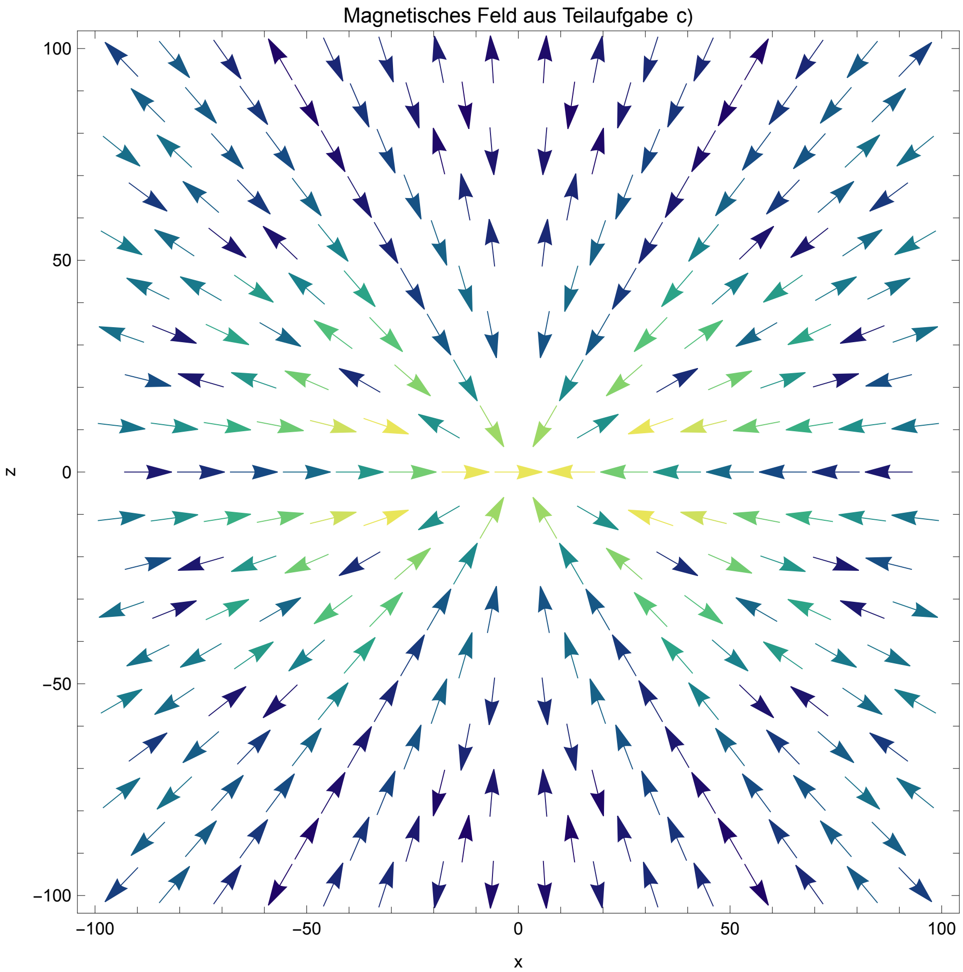
$$\text{VectorPlot}\left[\left\{-\frac{x\,z\,\text{Cos}\left[1-\sqrt{x^2+z^2}\right]}{2\pi\left(x^2+z^2\right)^{3/2}},\frac{\left(x^2\right)\text{Cos}\left[1-\sqrt{x^2+z^2}\right]}{2\pi\left(x^2+z^2\right)^{3/2}}\right\},\right.$$

```
 {x, -100, 100}, {z, -100, 100}, FrameLabel → {"x", "z"}, PlotRange → All,
 VectorColorFunction → "BlueGreenYellow",
 PlotLabel → "Elektrisches Feld aus Teilaufgabe f)", ImageSize → 500]
```

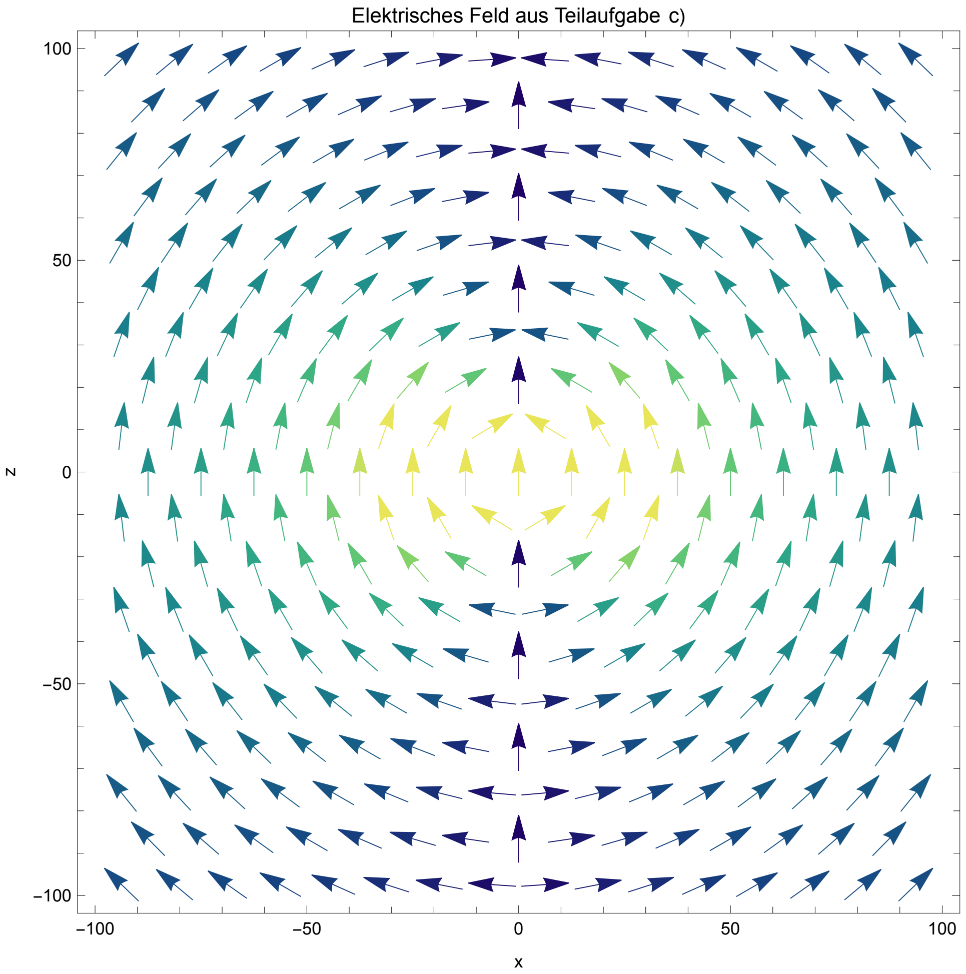
$$\text{VectorPlot}\left[\left\{\frac{x\left(x^2\right)\text{Cos}\left[1-\sqrt{x^2+z^2}\right]^2}{4\pi^2\left(x^2+z^2\right)^{5/2}},\frac{\left(x^2\right)z\,\text{Cos}\left[1-\sqrt{x^2+z^2}\right]^2}{4\pi^2\left(x^2+z^2\right)^{5/2}}\right\},\right.$$

```
 {x, -100, 100}, {z, -100, 100}, FrameLabel → {"x", "z"}, PlotRange → All,
 VectorColorFunction → "BlueGreenYellow",
 PlotLabel → "Poynting Vektor aus Teilaufgabe f)", ImageSize → 500]
```

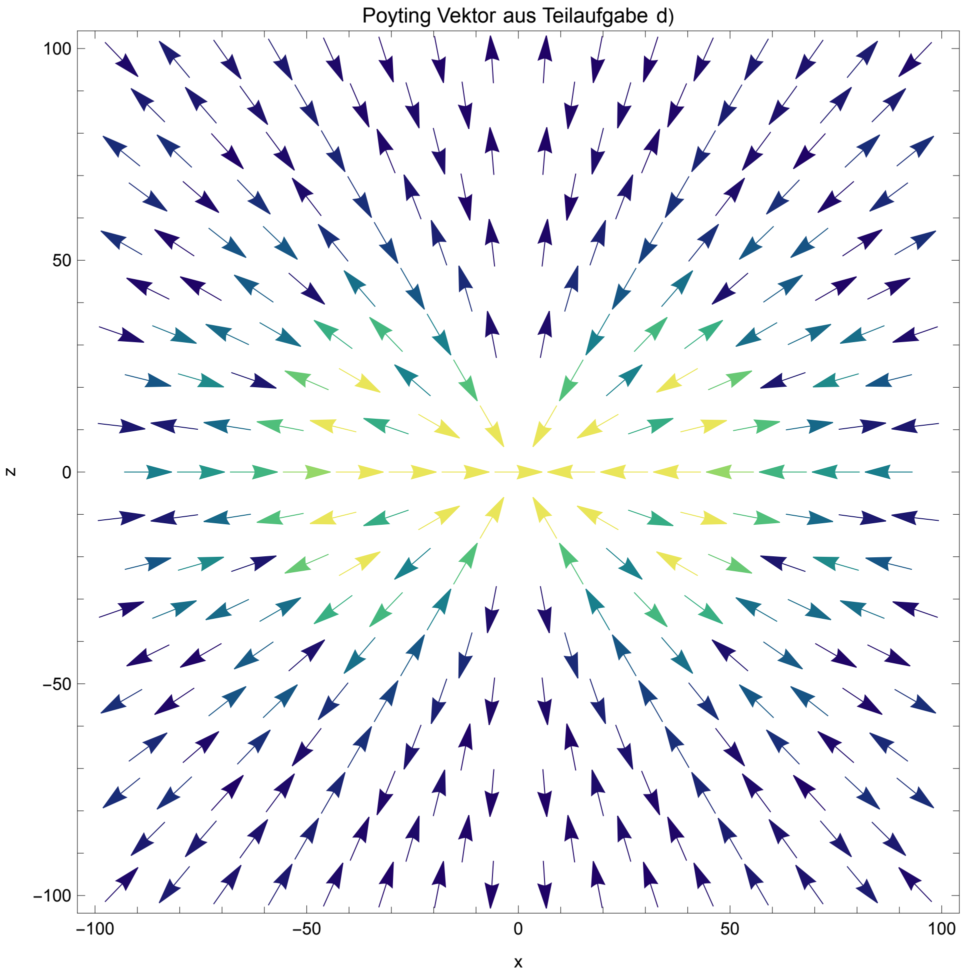
Out[132]=


Magnetisches Feld aus Teilaufgabe c)
100
50
0
−50
−100
z
−100
−50
0
50
100
x

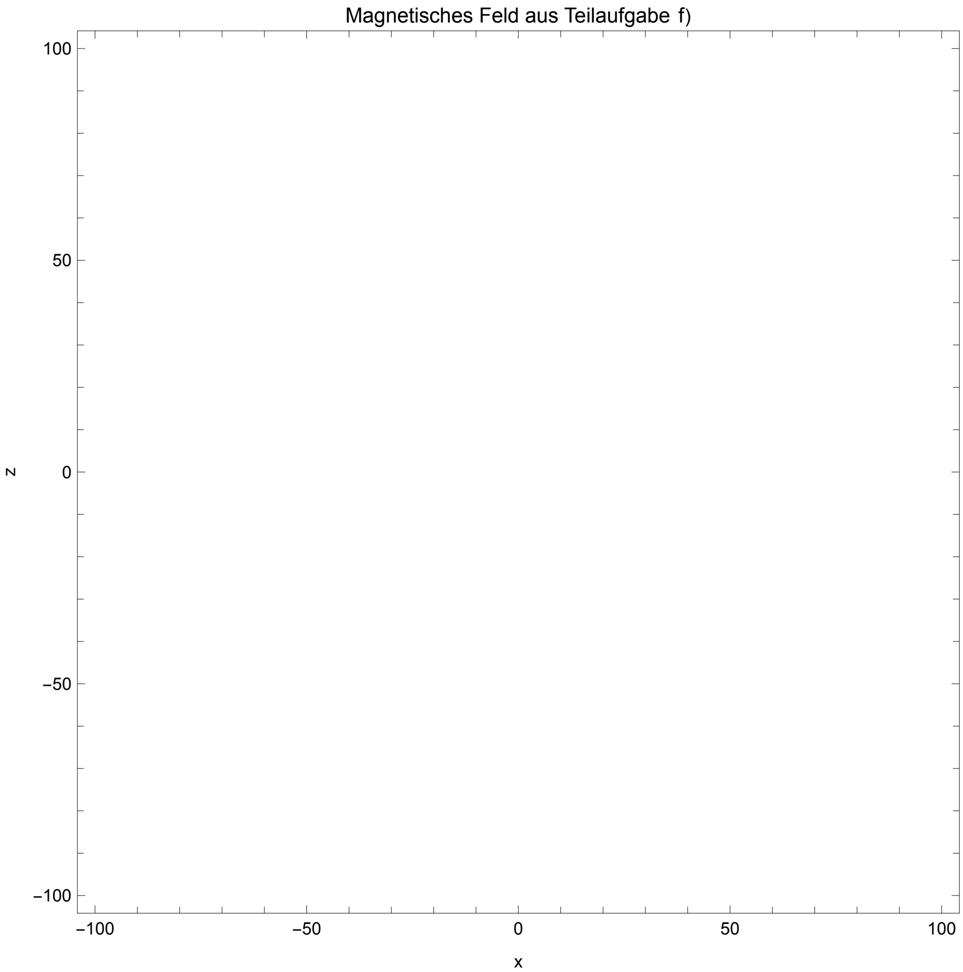
Out[133]=


Elektrisches Feld aus Teilaufgabe c)
100
50
0
−50
−100
z
−100
−50
0
50
100
x

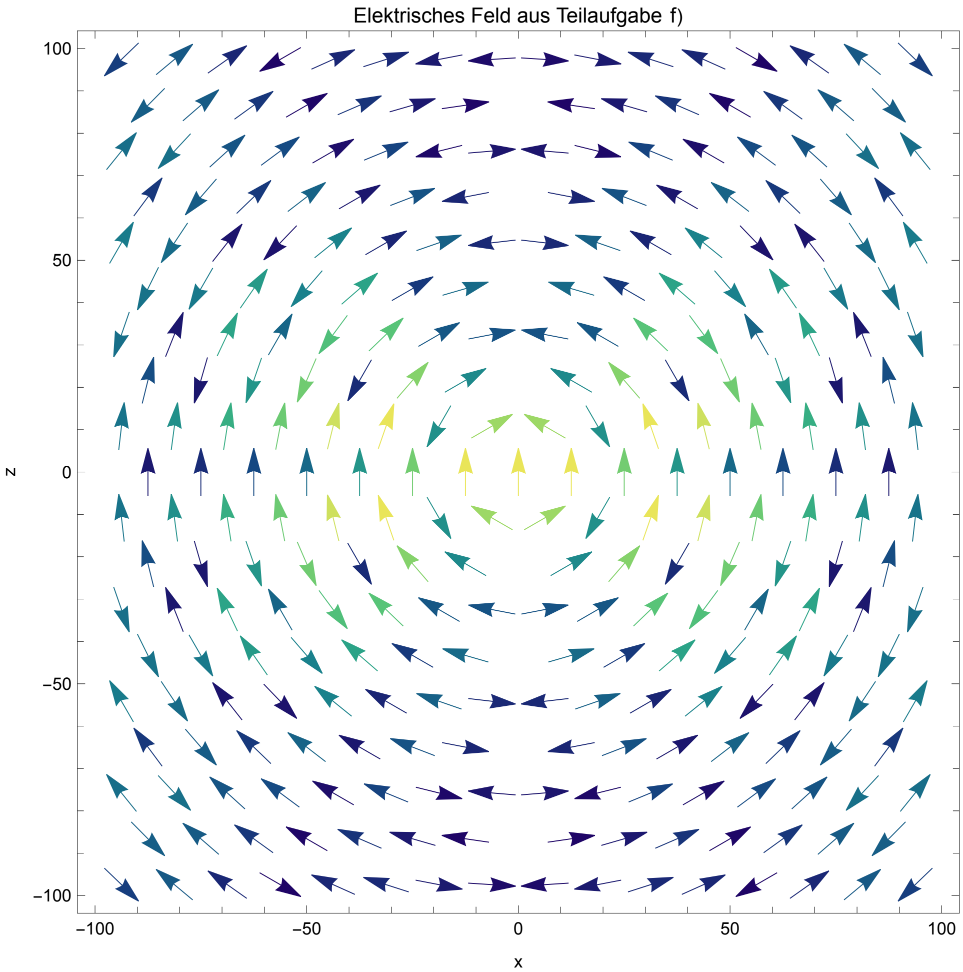
Out[134]=


Poyting Vektor aus Teilaufgabe d)
100
50
0
−50
−100
z
−100
−50
0
50
100
x

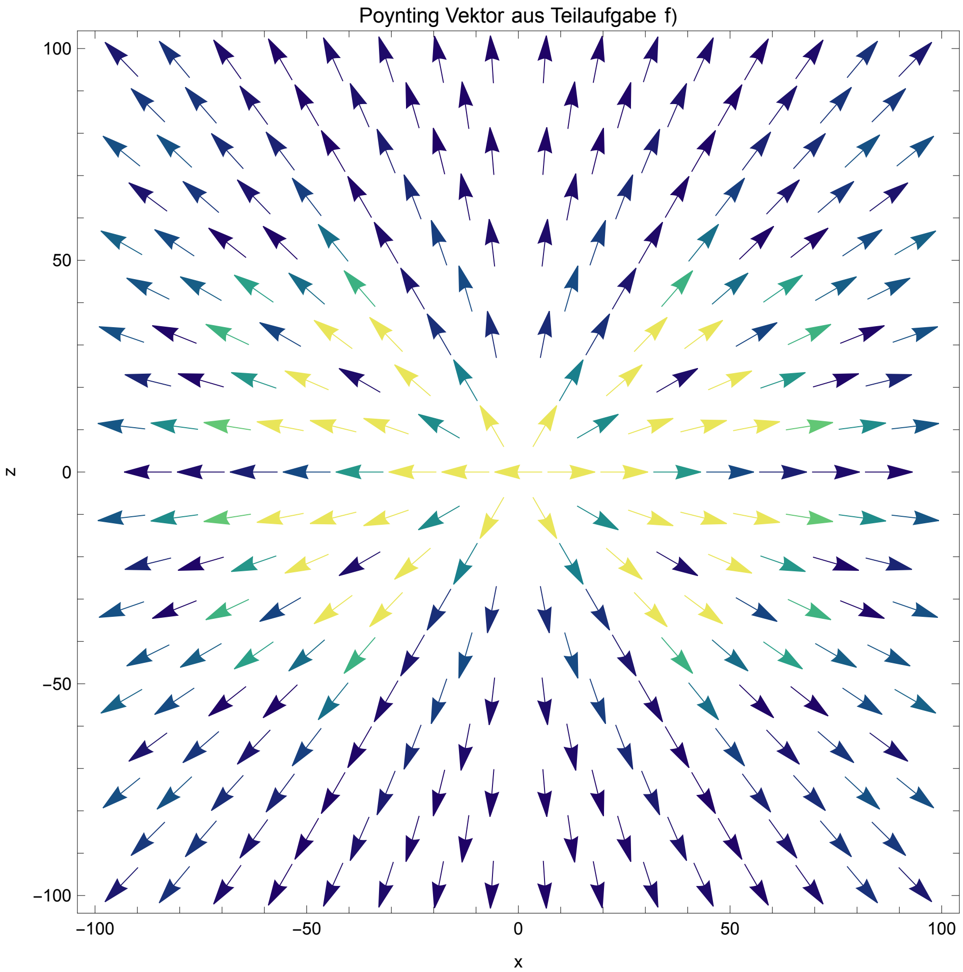
Out[135]=


Magnetisches Feld aus Teilaufgabe f)
100
50
0
−50
−100
z
−100
−50
0
50
100
x

Out[136]=


Elektrisches Feld aus Teilaufgabe f)
100
50
0
−50
−100
z
−100
−50
0
50
100
x

Out[137]=



Die abgestrahlten Leistungen aus den Teilaufgaben e) und f) unterscheidet sich nicht.