Statistik: Teilnehmer: 189

Schein	über 50	Punkte:	57
	" 45	"	94
mehr " als sein			
	42 – 44	"	1
	unter 42	"	94

Goldmedaille:		86	Punkte	(nur Klausur)
Silber " :		85	"	"
Bronze "	2x	78	"	"

1 a) $[X_j, L_k] = \sum_{m,n} \varepsilon_{kmn} [X_j, X_m P_n] = \sum_{m,n} \varepsilon_{kmn} X_m \underbrace{[X_j, P_n]}_{i\hbar \delta_{jn}}$

$$= \sum_m i\hbar\, \varepsilon_{kmj} X_m$$

analog $[P_j, L_k] = -i\hbar \sum_n \varepsilon_{kjn} P_n$

b) $[X_j X_j, L_k] = X_j [X_j, L_k] + [X_j, L_k] X_j$

$$\overset{a)}{=} 2i\hbar \sum_m \varepsilon_{jkm} X_j X_m$$

$$[\vec{X}^2, L_k] = \sum_{j=1}^{3} [X_j^2, L_k] = 2i\hbar \sum_{j,m} \varepsilon_{jkm} \underbrace{X_j X_m}_{\text{symm}} = 0$$

analog: $[\vec{x}\cdot\vec{p}, L_k] = 0$

c) Wegen $[\vec{L}, \vec{S}] = 0 \Rightarrow \vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}$

$2\vec{L}\cdot\vec{S} = 2S_3L_3 + S_+L_- + S_-L_+$ mit $S_1 = \frac{S_+ + S_-}{2}$

$S_2 = \frac{S_+ - S_-}{2i}$ usw

$\Rightarrow \vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2S_3L_3 + S_+L_- + S_-L_+$,

d) $\left.\begin{matrix} S_+ \\ L_+ \end{matrix}\right\} |l\,l\,s\,s\rangle = 0 \Rightarrow \vec{J}^2 |l\,l\,s\,s\rangle =$

$= \hbar^2 [l(l+1) + s(s+1) + 2ls] \cdot |l\,l\,s\,s\rangle$

e) $s = 1/2$:

$$|\psi\rangle = a_{\ell m} \, |\ell \;\; m-\tfrac{1}{2} \;\; s \, \tfrac{1}{2}\rangle + b_{\ell m} \, |\ell \;\; m+\tfrac{1}{2} \;\; s \; -\tfrac{1}{2}\rangle$$

$$\vec{J}^2 |\psi\rangle \overset{!}{=} \hbar^2 j(j+1) |\psi\rangle \overset{!}{=} \hbar^2 (\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2}) \, |\psi\rangle$$

$$\vec{J}^2 |\psi\rangle \overset{d)}{=} \hbar^2 \left[\ell(\ell+1) + m + \frac{1}{2} + \frac{b_{\ell m}}{a_{\ell m}} \sqrt{(\ell-m+\tfrac{1}{2})(\ell+m+\tfrac{1}{2})} \right] \cdot a_{\ell m} \cdot |\ell \;\; m-\tfrac{1}{2} \;\; \tfrac{1}{2} \;\; \tfrac{1}{2}\rangle$$

$$+ \hbar^2 \left[\ell(\ell+1) - m + \frac{1}{4} + \frac{a_{\ell m}}{b_{\ell m}} \sqrt{(\ell-m+\tfrac{1}{2})(\ell+m+\tfrac{1}{2})} \right] b_{\ell m} \cdot |\ell \;\; m+\tfrac{1}{2} \;\; \tfrac{1}{2} \;\; -\tfrac{1}{2}\rangle$$

$$\Rightarrow \quad \frac{b_{\ell m}}{a_{\ell m}} = \frac{\sqrt{\ell - m + 1/2}}{\sqrt{\ell + m + 1/2}}$$

2) a) $$H\psi = \left[-\frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

$$\Rightarrow \quad E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 q^2}{2m} + V_0 \Rightarrow k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$q = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

Stetigkeit von $\psi(x)$: $\quad 1 + R = T$

" " $\psi'(x)$: $\quad k(1-R) = qT$

$$\Rightarrow R = \frac{k-q}{k+q}$$

$$T = 1 + R = \frac{2k}{k+q}$$

b) $$|R|^2 = \left(\frac{1 - \sqrt{1 - V_0/E}}{1 + \sqrt{1 - V_0/E}} \right)^2 \; ; \quad \frac{q}{k} |T|^2 = \frac{4 \sqrt{1 - V_0/E}}{\left(1 + \sqrt{1 - V_0/E}\right)^2}$$

c) $$\psi_R = T e^{-\kappa x}$$: Analog zu a), b)

$$R = \frac{k - i\kappa}{k + i\kappa} \; ; \quad T = \frac{2k}{k + i\kappa}$$

d) $$|R|^2 = \frac{|k - i\kappa|}{|k + i\kappa|} = 1 \; ; \quad I(\kappa) = \int_0^\infty dx \, e^{-2\kappa x} = \frac{1}{2\kappa}$$

$$\int_0^\infty dx \cdot x e^{-2\kappa x} = -\frac{1}{2} \frac{d}{d\kappa} I(\kappa) = \frac{1}{2\kappa} I(\kappa) \Rightarrow d = \frac{1}{2\kappa}$$

3. a) $$\left[X, e^{\frac{i}{\hbar} l\cdot P}\right] = i\hbar \frac{\partial}{\partial P} e^{\frac{i}{\hbar} l P} = (-l)\, e^{\frac{i}{\hbar} l P}$$

alternativ: CBH-Formel

b) $$a|l\rangle = a\, e^{i l P/\hbar} |0\rangle =$$
$$= \frac{1}{\sqrt{2}}\left(\frac{X}{x_0} + i\,\frac{x_0 P}{\hbar}\right) e^{i l P/\hbar} |0\rangle$$
$$= \frac{\left[X, e^{i l P/\hbar}\right]}{\sqrt{2}\, x_0} |0\rangle + e^{i l P/\hbar} \underbrace{a|0\rangle}_{=0}$$

$$= \frac{-\ell}{\sqrt{2}\, x_0} |\ell\rangle$$

c)

$$\langle \ell | P | \ell \rangle = \langle 0 | e^{-i\ell P/\hbar} P e^{\frac{i\ell P}{\hbar}} |0\rangle = \underbrace{\langle 0 | P | 0 \rangle}_{0} =$$

$$\langle \ell | X | \ell \rangle = \langle 0 | e^{-i\ell P/\hbar} \underbrace{X e^{i\ell P/\hbar}}_{a)} |0\rangle =$$

$$= \langle 0 | e^{-i\ell P/\hbar} \underbrace{[X, e^{i\ell P/\hbar}]}_{\text{in } a)} |0\rangle + \underbrace{\langle 0 | X | 0 \rangle}_{=0}$$

$$= -\ell \, \langle 0 | 0 \rangle = -\ell$$

d) Analog: $\langle l|p^2|l\rangle = \langle 0|p^2|0\rangle = \frac{\hbar^2}{2x_0^2}$

$$\langle l|x^2|l\rangle = \langle 0|(x-l)\, e^{-ilp/\hbar}\, e^{ilp/\hbar}(x-l)|0\rangle$$

$$= \langle 0|x^2 + l^2 - 2lx|0\rangle =$$

$$= \frac{x_0^2}{2} + l^2$$

$$\Delta p = \frac{\hbar}{\sqrt{2}\,x_0} \quad ; \quad \Delta x = \sqrt{\frac{x_0^2}{2} + l^2 - \underbrace{l^2}_{\langle x\rangle^2}} = \frac{x_0}{\sqrt{2}}$$

$$\Delta p \cdot \Delta x = \frac{\hbar}{2}$$

e) $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$; $c_n = \langle \ell | n\rangle$

$$c_n = \frac{1}{\sqrt{n!}}\langle 0| a^n |\ell\rangle \overset{a)}{=} \frac{1}{\sqrt{n!}}\left(\frac{-\ell}{\sqrt{2}\,x_0}\right)^n \underline{\langle 0|\ell\rangle}$$

Given $\quad P = -\frac{i\hbar}{x_0}\,\frac{a-a^+}{\sqrt{2}}$; $\xi = \frac{\ell}{\sqrt{2}\,x_0}$

$$\langle 0|\ell\rangle = \langle 0|\, e^{\xi(a-a^+)}|0\rangle = \ldots$$

$$\overset{\text{Hinweis}}{=} \langle 0|e^{-\xi a^+}\, e^{\xi a}\, \underline{e^{-\xi^2[a,a^+]/2}}|0\rangle = e^{-\xi/2}\langle 0|0\rangle$$

$$= e^{-\xi^2/2} \Rightarrow |\ell\rangle = e^{-\xi^2/2}\sum_{n=0}^{\infty}\frac{(-\xi)^n}{\sqrt{n!}}|n\rangle$$

Aufgabe 4: wie Übungsblatt

5) a) Eigenwerte $E_\pm = \pm \frac{\hbar\omega}{2}$; $\hat{H} = \frac{\hbar\omega}{2} M$

b) $M = \cos(\Omega t)\,\sigma_2 + \sin(\Omega t)\,\sigma_3$

c) $$M = \left(1\cos\frac{\Omega t}{2} - i\vec{n}\vec{\sigma}\sin\frac{\Omega t}{2}\right)\sigma_2\left(1\cos\frac{\Omega t}{2} + i\vec{n}\vec{\sigma}\sin\frac{\Omega t}{2}\right)$$

$$= \sigma_2\cdot\cos^2\frac{\Omega t}{2} + i[\sigma_2, \vec{n}\vec{\sigma}]\,\underbrace{\cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2}}_{\frac{1}{2}\sin(\Omega t)}$$

$$+ (\vec{n}\vec{\sigma})\sigma_2\cdot(\vec{n}\vec{\sigma})\cdot\sin^2\frac{\Omega t}{2}$$

$$[\vec{n}\vec{\sigma}, \sigma_2] = \sum_{j=1}^{3} n_j [\sigma_j, \sigma_2] = 2i \sum_{k=1}^{3} n_j \, \varepsilon_{j2k} \sigma_k$$

Vergleich mit b) $\quad n_1 = 1$ und $n_3 = 0$; $\vec{n}^2 = 1 \Rightarrow n_2 = 0$

$$(\vec{n}\vec{\sigma})\sigma_2(\vec{n}\vec{\sigma}) = \sigma_1\sigma_2\sigma_1 = i\,\sigma_3\sigma_1 = -\sigma_2 = -\sigma_2$$

Koeff von σ_2: $\cos^2\frac{\Omega t}{2} - \sin^2\frac{\Omega t}{2} = \cos \Omega t$

d) $$\frac{\partial \chi}{\partial t} = i\frac{\Omega}{2}(\vec{n}\vec{\sigma}) \cdot e^{i\Omega t\,\vec{n}\vec{\sigma}/2}\psi + e^{i\Omega t\,\vec{n}\vec{\sigma}/2}\frac{\partial \psi}{\partial t}$$

$$= i\frac{\Omega}{2}\,\vec{n}\vec{\sigma}\,\chi + e^{i\Omega t\,\vec{n}\vec{\sigma}/2}\frac{\partial \psi}{\partial t}$$

und $i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi = \frac{\hbar\omega}{2}\, e^{-i\Omega t\,\vec{n}\vec{\sigma}/2}\,\sigma_2\,\chi \Longrightarrow$

$$i\hbar \frac{\partial \chi}{\partial t} = \left(-\frac{\hbar\Omega}{2}\sigma_1 + \frac{\hbar\omega}{2}\sigma_2\right)\chi$$

$$\chi(t) = e^{i\left(\frac{\Omega t}{2}\sigma_1 - \frac{\omega t}{2}\sigma_2\right)}\chi(0) \; ; \quad \chi(0) = \psi(0) ;$$

$$\psi(t) = e^{-i\frac{\Omega t}{2}\sigma_1} e^{i\left(\frac{\Omega t}{2}\sigma_1 - \frac{\omega t}{2}\sigma_2\right)}\psi(0)$$