Musterlösung Theorie D, Blatt 8

Aufgabe 1: We first consider in general:

$$\left(\frac{\partial g}{\partial x}\right)_{y,z} = \left(\frac{\partial \theta}{\partial x}\right)_{y,z}\left(\frac{\partial f}{\partial \theta}\right)_{\phi,r} + \left(\frac{\partial \phi}{\partial x}\right)_{y,z}\left(\frac{\partial f}{\partial \phi}\right)_{\theta,r} + \left(\frac{\partial r}{\partial x}\right)_{y,z}\left(\frac{\partial f}{\partial r}\right)_{\theta,\phi},$$

where $g = g(x,y,z)$, and $f = f(r(x,y,z),\theta(x,y,z),\phi(x,y,z))$.

$r = \sqrt{(x^2+y^2+z^2)}$,

$\frac{y}{x} = \tan\phi$,

$\mathrm{Cos}\,\theta = \frac{z}{r} = \frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}$.

$$\left(\frac{\partial \mathrm{Cos}\theta}{\partial x}\right)_{y,z} = -\left(\frac{\partial \theta}{\partial x}\right)_{y,z}\mathrm{Sin}\theta = -\frac{1}{2}\frac{z\ 2x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \Rightarrow \left(\frac{\partial \theta}{\partial x}\right)_{y,z} = +\frac{r^2\mathrm{Cos}\theta\mathrm{Sin}\theta\mathrm{Cos}\phi}{r^3 Sin\theta},$$

$$\left(\frac{\partial \tan\theta}{\partial x}\right)_{y,z} = \frac{1}{\mathrm{Cos}^2\phi}\left(\frac{\partial \phi}{\partial x}\right)_{y,z} = -\frac{y}{x^2} \Rightarrow \left(\frac{\partial \phi}{\partial x}\right)_{y,z} = -\frac{r\mathrm{Sin}\theta\mathrm{Sin}\phi\mathrm{Cos}^2\phi}{r^2\mathrm{Sin}^2\theta\mathrm{Cos}^2\phi},$$

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{1}{2}\frac{2x}{r} \Rightarrow \left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{r\mathrm{Sin}\theta\mathrm{Cos}\phi}{r}.$$

Analogous expressions can also be derived for the derivatives of r,θ,ϕ with respect to y and z. The momentum operator

$$\begin{aligned} i\mathrm{h}p_x &= \frac{\partial}{\partial x} \\ &= \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} + \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \sin\theta\cos\phi\frac{\partial}{\partial r} \end{aligned}$$

And similar expressions for the p_y, p_z. Inserting these expressions into:

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = yp_z - zp_y,$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = zp_x - xp_z,$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = xp_y - yp_x.$$

we get the equations which we must square and sum up to get the expression for L^2:

$$L^2 = -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \frac{\mathrm{Cos}\theta}{\mathrm{Sin}\theta}\frac{\partial}{\partial\theta} + \frac{1}{\mathrm{Sin}^2\theta}\frac{\partial^2}{\partial\phi^2} \right).$$

Aufgabe 2:

a) Expand F in a Power-Series in B (this is the definition anyway) and evaluate term by term. Finally resum
b) Use $S^\dagger(a) = S(-a)$
c) Compute $[\hat{x}, S(\lambda)] = \lambda S(\lambda)$, using the commutation relation of x and p and the result of (a)
d) From $\hat{x}S(\lambda)|x\rangle = S(\lambda)(\hat{x}+\lambda)|x\rangle = S(\lambda)(x+\lambda)|x\rangle = (x+\lambda)S(\lambda)|x\rangle$
it follows that $S(\lambda)|x\rangle$ is an Eigenvector of $\hat{x}$ with Eigenvalue $x+\lambda$. According to the definition of $|x\rangle$ this means that $S(\lambda)|x\rangle : |x+\lambda\rangle$ and without loss of generality we can define the phase as unity relative to some chosen Eigenvector $|0\rangle$. Then $|x\rangle = S(x)|0\rangle$ and $S(\lambda)|x\rangle = |x+\lambda\rangle$. This means that S is the translation operator.

Aufgabe 3:

We are using

$$|\psi\rangle = e^{i\frac{\chi}{2}}\left(\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\left|\begin{pmatrix}1\\0\end{pmatrix}\right\rangle + \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\left|\begin{pmatrix}0\\1\end{pmatrix}\right\rangle\right)$$

Note: the unit vectors have been interchanged from the exercise for consistency.

Note $\sigma_i^2 = 1$, therefore $\langle\psi|\sigma_i^2|\psi\rangle = 1$ for i=x,y,z.

$$\langle\psi|\sigma_z|\psi\rangle = \langle\psi|\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}|\psi\rangle = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$

$$\begin{aligned}\langle\psi|\sigma_x|\psi\rangle = \langle\psi|\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}|\psi\rangle &= \langle\psi|e^{i\frac{\chi}{2}}\left(e^{i\frac{\varphi}{2}}\cos\frac{\theta}{2}\left|\begin{pmatrix}0\\1\end{pmatrix}\right\rangle + e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\right)\left|\begin{pmatrix}1\\0\end{pmatrix}\right\rangle \\ &= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i\phi} + e^{-i\phi}\right) \\ &= \sin\theta\cos\phi\end{aligned}$$

$$\langle\psi|\sigma_y|\psi\rangle = \langle\psi|\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}|\psi\rangle = \sin\theta\sin\varphi$$

Then use: $(\Delta\sigma_x\Delta\sigma_y)^2 = 1 - \sin^2\theta + \sin^4\theta\sin^2\varphi\cos^2\phi > 1 - \sin^2\varphi = \cos^2\theta$. This means, that even for a “pure state” you cannot pick the axis at which you measure at random. Only for the special values where the last term before the inequality is zero is the uncertainty relationship met exactly.