(S. Sakurai 1.29, 1.30) Translationsoperator

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2.7.
$$[x_i, G(\vec p)] = i\hbar \frac{\partial G}{\partial p_i} \quad ; \quad [p_i, F(\vec x)] = -i\hbar \frac{\partial F}{\partial x_i}$$

$$G(\vec p) = \sum_{klm} \alpha_{klm}\, p_1^k\, p_2^l\, p_3^m$$

$\rightarrow [x_i, p_i^k p_j^l p_h^m] = ?$ $\rightarrow$ Induktion

$[x_i, p_i^k] = ?$ $\quad [x_i, p_i] = i\hbar$

$$[x_i, p_i^2] = x_i p_i^2 - p_i^2 x_i = (p_i x_i + [x_i, p_i])p_i - p_i^2 x_i$$
$$= p_i [x_i, p_i] + i\hbar p_i = 2 i\hbar p_i$$
$$[x_i, p_i^3] = (p_i x_i + [x_i, p_i])\, p_i^2 - p_i^3 x_i$$
$$= p_i [x_i, p_i^2] + i\hbar p_i^2 = 3 i\hbar p_i^2$$

Vermutung $\boxed{[x_i, p_i^k] = k\, i\hbar\, p_i^{k-1}}$ I.A.

$[x_i, p_i^{k+1}] = (k+1)\, i\hbar\, p_i^k$? I.V.
$$x_i p_i^{k+1} - p_i^{k+1} x_i = (p_i x_i + i\hbar)\, p_i^k - p_i p_i^k x_i$$
$$= p_i \underbrace{[x_i, p_i^k]}_{= k\, i\hbar\, p_i^{k-1}} + i\hbar p_i^k = (k+1)\, i\hbar\, p_i^k \checkmark$$

$\Rightarrow$
$$[x_i, p_i^k p_j^l p_h^m] = \underbrace{p_j^l p_h^m}\, [x_i, p_i^k]$$
$$[x_i, p_j] = [x_i, p_h] = 0$$
$$= i\hbar\, p_j^l p_h^m\, k\, p_i^{k-1} = i\hbar \left(\frac{\partial}{\partial p_i} \left(p_j^l p_k^m p_i^k \right) \right)$$

$$\Rightarrow [x_i, \alpha_{klm}\, p_i^k p_j^l p_n^m] \quad \dots$$

$$\Rightarrow \boxed{[x_i, G(\vec{p})] = i\hbar \frac{\partial G(\vec{p})}{\partial p_i}}$$

$[p_i, F(\vec{x})]$ analog

$$F(\vec{x}) = \sum_{klm} \beta_{klm}\, x_i^k x_j^l x_n^m$$

etc. $\rightarrow$ aber $[p_i, x_i] = -i\hbar$

Vorzeichen zieht sich durch Beweis für $[x_i, p_i^k]$

$$\rightsquigarrow \boxed{[p_i, F(\vec{x})] = -i\hbar \frac{\partial F(\vec{x})}{\partial x_i}}$$

Damit betrachte $T(\vec{l}) = e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{l}}$

(Translation um $\vec{l}$). $\vec{l}$ = parameter

$\vec{p}$ = Impulsoperator.

$$[x_i, T(\vec{l})] = ?$$

$$(5.0) \rightarrow \quad = i\hbar \frac{\partial T(\vec{l})}{\partial p_i} = \underbrace{i\hbar\left(-\frac{i}{\hbar}\right.}_{=1}\left. l_i\right) e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{l}}$$

$$\boxed{[x_i, T(\vec{l})] = l_i T(\vec{l})}$$

allgemeiner Induktionsbeweis

z.z. $[A,B^n] = n\,B^{n-1}[A,B]$, $[A,[A,B]] = [B,[A,B]] = 0$

Vermutung: $[A,B^2] = AB^2 - B^2A$

$= B[A,B] + [A,B]B = 2B[A,B]$

$[A,B^3] = AB^3 - B^3A = B^2[A,B] + [A,B^2]B$

$= B^2[A,B] + 2B[A,B]B$ (s.o.)

$= 3B^2[A,B]$ $\longrightarrow$ Verallgemeinerung auf Formel oben

Induktion:

$[A,B^{n+1}] = AB^{n+1} - B^{n+1}A$

$= B^n[A,B] + [A,B^n]B$

$= B^n[A,B] + nB^{n-1}B = (n+1)B^n[A,B]$ ✓

insbesondere für $[x_i, p_j] = i\hbar\delta_{ij}$

$[x_i, p_i^n] = np_i^{n-1}\, i\hbar$ $\quad [\underbrace{[x_i,p_i]}_{i\hbar}, p_i] = 0$

Verschobener Erwartungswert $\langle \vec{x} \rangle' = ?$

bei gegebenem EW $\langle \vec{x} \rangle$.

$$\langle \vec{x} \rangle = \langle \alpha | \vec{x} | \alpha \rangle$$

$$\langle \vec{x} \rangle' = \langle \alpha | T^+ \vec{x}\, T | \alpha \rangle$$

$$\rightarrow \langle \vec{x} \rangle' - \langle \vec{x} \rangle = \langle \alpha | \underset{\uparrow\, 1 = T^+T}{\vec{x}} - T^+ \vec{x}\, T | \alpha \rangle$$

$$= \langle \alpha | T^+ [\vec{x}, T] | \alpha \rangle$$

$$= \langle \alpha | T^+(\vec{\ell}) [\vec{x}, T(\vec{\ell})] | \alpha \rangle$$

nur für $x_i \rightarrow$ s.o.

$$\langle x_i \rangle' - \langle x_i \rangle = \langle \alpha | T^+(\vec{\ell}) [x_i, T(\vec{\ell})] | \alpha \rangle$$

$$= \langle \alpha | T^+ \ell_i T | \alpha \rangle = \ell_i \langle \alpha | \alpha \rangle = \ell_i$$

$$\Rightarrow \underline{\langle \vec{x} \rangle' = \langle \vec{x} \rangle + \vec{\ell}}$$

Drehmatrizen ⇔ Darstellungen des Drehimpulses in 3 Dimensionen.

Vorlesung: aus Standarddrehmatrizen, entwickelt für $\varphi \to 0$: Generatoren

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad G_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad G_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

a) Berechne die Kommutatoren u. zeige, daß G_i die Drehimpulsalgebra erfüllen

$$G_x G_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad G_y G_x = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$G_y G_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad G_z G_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$G_z G_x = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad G_x G_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad [G_x, G_y] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i G_z$$

$$[G_y, G_z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i G_x$$

$$[G_z, G_x] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = i G_y$$

$$\Rightarrow \quad [G_i, G_j] = i \varepsilon_{ijk} G_k$$

$\Rightarrow$ die G_i sind mögliche 3-dimensionale Darstellung der Drehgruppe.

üblicherweise erhält man in der J_z - Darstellung die Darstellung (s. Theo D):

$$l_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \quad l_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$l_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

diese müssten unitär äquivalent sein!

$\rightarrow$

b) finde $\mathcal{U}$

$\rightarrow$ l_z = diagonal ; EW 1, 0, −1

$\rightsquigarrow$ diagonalisiere $G_z \rightarrow$ EV $\rightarrow \mathcal{U}$

$G_z \vec{n} = \lambda \vec{n} \rightarrow$

$$\det \begin{pmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \overset{!}{=} 0$$

$$= -\lambda^3 - (+i \cdot i(+\lambda)) = -\lambda^3 + \lambda = 0$$

$$= \lambda(1-\lambda^2) = 0 \Rightarrow \lambda = 0, \pm 1$$

(wie erwartet).

EV:

$\underline{\lambda=0}$ $\quad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u} = 0 \cdot \vec{u} = \vec{0}$

$\Rightarrow \quad \begin{pmatrix} -iy \\ ix \\ 0 \end{pmatrix} = \vec{0} \Rightarrow x = y = 0 \Rightarrow \vec{u}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda=1}$ $\quad \begin{pmatrix} -iy \\ ix \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x = -iy \,, z = 0$

$i(-i)y = y$ ✓

$\Rightarrow \quad y \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \vec{u} \Rightarrow \quad \vec{u}_1 =$

$\vec{u}\vec{u}^* = 1+1 = 2 \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$

$\lambda=-1$ $\quad \begin{pmatrix} -iy \\ ix \\ 0 \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \quad z = 0$

$+iy = +x$

$\Rightarrow \quad i(iy) = -y$ ✓

$y \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = \vec{u} \Rightarrow \quad \vec{u}_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ unitäre Transformation

$$(\vec{u}_1, \vec{u}_0, \vec{u}_{-1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & i \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \mathcal{U}$$

check:

$$\mathcal{U}^*\mathcal{U} = \frac{1}{2}\begin{pmatrix} i & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -i & 1 & 0 \end{pmatrix}\begin{pmatrix} -i & 0 & i \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

→ diagonalisieren:

$$\mathcal{U}\, G_z\, \mathcal{U}^* = \frac{1}{2}\begin{pmatrix} i\alpha^* & 1\alpha^* & 0 \\ 0 & 0 & \sqrt{2} \\ -i\beta^* & 1\beta^* & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -i\alpha & 0 & i\beta \\ 1\alpha & 0 & 1\beta \\ 0\alpha & \sqrt{2} & 0\beta \end{pmatrix}$$

$$= \frac{1}{2}\left(\text{— '' —} \right)\begin{pmatrix} -i\alpha & 0 & -i\beta \\ 1\alpha & 0 & -1\beta \\ 0\alpha & 0 & 0\beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = l_z \quad \checkmark$$

$$\mathcal{U}\, G_x\, \mathcal{U}^* = \frac{1}{2}\begin{pmatrix} i\alpha^* & 1\alpha^* & 0 \\ 0 & 0 & \sqrt{2} \\ -i\beta^* & 1\beta^* & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} -i\alpha & 0 & i\beta \\ 1\alpha & 0 & 1\beta \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{1}{2}\left(\text{''} \right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\sqrt{2} & 0 \\ i\alpha & 0 & i\beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{2}\alpha^* & 0 \\ i\sqrt{2}\alpha & 0 & i\sqrt{2}\beta \\ 0 & -i\sqrt{2}\beta^* & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i\alpha^* & 0 \\ i\alpha & 0 & i\beta \\ 0 & -i\beta^* & 0 \end{pmatrix}$$

→ use phases in u_1 → $\vec{u}_1 = \alpha$

$\vec{u}_4 = \beta$

with $\alpha\alpha^* = \beta\beta^* = 1$

→ $\alpha = \beta = -i$?

$$\mathcal{U}\, G_y\, \mathcal{U}^* = \frac{1}{2}\begin{pmatrix} i\alpha^* & \alpha^* & 0 \\ 0 & 0 & \sqrt{2} \\ -i\beta^* & \beta^* & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} -i\alpha & 0 & i\beta \\ \alpha & 0 & \beta \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} - \text{"} - \end{pmatrix}\begin{pmatrix} 0 & i\sqrt{2} & 0 \\ 0 & 0 & 0 \\ -\alpha & 0 & \beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\sqrt{2}\alpha^* & 0 \\ -\sqrt{2}\alpha & 0 & \sqrt{2}\beta \\ 0 & \sqrt{2}\beta^* & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\alpha^* & 0 \\ -\alpha & 0 & \beta \\ 0 & \beta^* & 0 \end{pmatrix} \underset{\substack{\uparrow \\ \alpha = \beta = -i}}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$\Rightarrow$ phases chosen as $\underline{\alpha = \beta = -i}$

$$\Rightarrow \quad \mathcal{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

transforms G_i to corresponding l_i ✓

2.7.

$$R(\vec{\omega})\; R(-\vec{\omega}) = \mathbb{1}$$

$$\rightsquigarrow \quad R(-\vec{\omega}) = R^T(\vec{\omega}) = R(\vec{\omega})^{-1} \quad ?$$

$$R(\vec{\omega})_{ij} = \frac{\omega_i \omega_j}{\omega^2} + \left(\delta_{ij} - \frac{\omega_i\omega_j}{\omega^2}\right)\cos\omega + \frac{\sin\omega}{\omega}\underbrace{\varepsilon_{ikj}\,\omega_k}_{-\varepsilon_{ijk}\,\omega_k}$$

$$\varepsilon_{ijk}\,\omega_k \;\; k \to m, \; k \to m$$

$$R^T(\vec{\omega})_{jk} = R(\vec{\omega})_{kj}$$

$$= \frac{\omega_k\omega_j}{\omega^2} + \left(\delta_{kj} - \frac{\omega_k\omega_j}{\omega^2}\right)\cos\omega + \frac{\sin\omega}{\omega}\left(-\varepsilon_{kjl}\,\omega_l\right)$$

$$= R(-\vec{\omega})_{jk} \quad \Rightarrow \quad \underline{R(-\vec{\omega}) = R^T(\vec{\omega})}$$

now check $R(\vec{\omega})\,R(\vec{\omega}) = \mathbb{1}$?

$$R(\vec{\omega})\,R(-\vec{\omega})_{ik} = R(\vec{\omega})_{ij}\,R(-\vec{\omega})_{jk}$$

$$= \frac{\omega_i\omega_j\omega_j\omega_k}{\omega^4} + \left(\frac{2\omega_i\omega_k}{\omega^2} - \frac{2\omega_i\omega_k}{\omega^2}\right)\cos\omega$$

$$+ \left(\delta_{ik} - \frac{2\omega_i\omega_k}{\omega^2} + \frac{\omega_i\omega_k}{\omega^2}\right)\cos^2\omega$$

$$+ \underbrace{\varepsilon_{ijm}\,\varepsilon_{kjl}}_{=\varepsilon_{mij}\,\varepsilon_{jlk}}\,\omega_m\omega_l\,\frac{\sin^2\omega}{\omega}$$

(all terms symmetric × antisymmetric $= 0$) !

$$= \frac{\omega_i\omega_k}{\omega^2} + \left(\delta_{ik} - \frac{\omega_i\omega_k}{\omega^2}\right)\cos^2\omega$$

$$+ \left(\delta_{ml}\,\delta_{ik} - \delta_{mk}\,\delta_{il}\right)\omega_m\omega_l\,\frac{\sin^2\omega}{\omega^2}$$

$$= \frac{\omega_i \omega_k}{\omega^2} + \left(\delta_{ik} - \frac{\omega_i \omega_k}{\omega^2} \right) \cos^2 \omega$$

$$+ \left(\omega^2 \delta_{ik} - \omega_k \omega_i \right) \frac{\sin^2 \omega}{\omega^2}$$

$$= \frac{\omega_i \omega_k}{\omega^2} \left(1 - \cos^2 \omega - \sin^2 \omega \right) + \delta_{ik} \left(\cos^2 \omega + \sin^2 \omega \right)$$

$$= \delta_{ik}$$

$$\Rightarrow \boxed{R(\vec{\omega})\, R(-\omega) = \mathbb{1}} \quad \checkmark$$

$U^+ \vec{r}\, U = ?$ ($\vec{r}$ Operator!)

mit $U = e^{-\frac{i}{\hbar}\vec{\omega}\cdot\vec{L}}$ in ark rep!

use $e^{B} A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} A_n$

$A_0 = A \; ; \; A_n = [B, A_{n-1}]$

$\Rightarrow B \equiv \frac{i}{\hbar}\vec{\omega}\cdot\vec{L} \, , \; A = \vec{r}$

$(A_1)_i = ([B, A])_i = \frac{i}{\hbar}\omega_j [L_j, r_i]$

$= \frac{i}{\hbar}\omega_j [\varepsilon_{jkl} r_k p_l, r_i]$

$= \frac{i}{\hbar}\omega_j r_k \varepsilon_{jkl} [p_l, r_i]$

$= \frac{i}{\hbar}\omega_j r_k \varepsilon_{jkl} (-i\hbar)\delta_{li}$

$= \omega_j r_k \varepsilon_{jki} = \varepsilon_{ijk}\omega_j r_k = (\vec{\omega}\times\vec{r})_i$

$(A_2)_i = \left([\frac{i}{\hbar}\omega_j L_j , (\vec{\omega}\times\vec{r})_i] \right)$

$= \varepsilon_{ikl}\,\omega_k \underbrace{\frac{i}{\hbar}\omega_j [L_j, r_l]}_{(\vec{\omega}\times\vec{r})_l}$

$= \left(\vec{\omega}\times(\vec{\omega}\times\vec{r})\right)_i$

$= \varepsilon_{ijk}\,\omega_j\, \varepsilon_{klm}\,\omega_l r_m = \varepsilon_{ijk}\varepsilon_{klm}\omega_j\omega_l r_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\omega_j\omega_l r_m$

$$= \vec{\omega}\cdot\vec{r}\ \omega_i \quad - \quad \vec{\omega}^2 r_i$$

$$= \left(\vec{\omega}\cdot\vec{r}\ \vec{\omega} - \vec{\omega}^2\vec{r}\right)_i \quad = \quad (A_2)_i$$

$$(A_3)_i = [B, (A_2)_i]$$

$$= \frac{i}{\hbar}\omega_j [L_j, \vec{\omega}\cdot\vec{r}\ \omega_i - \vec{\omega}^2 r_i]$$

$$= \frac{i}{\hbar}\omega_j [L_j, \omega_k r_k \omega_i] - \frac{i}{\hbar}\omega_j [L_j, \vec{\omega}^2 r_i]$$

$$= \omega_k\omega_i\ \frac{i}{\hbar}\omega_j [L_j, r_k] - \vec{\omega}^2\ \frac{i}{\hbar}\omega_j [L_j, r_i]$$

$$= \omega_k\omega_i\ (\vec{\omega}\times\vec{r})_k \quad - \quad \vec{\omega}^2\ (\vec{\omega}\times\vec{r})_i$$

$$= \underbrace{\vec{\omega}\cdot(\vec{\omega}\times\vec{r})}_{=0}\ \omega_i \quad - \quad \vec{\omega}^2 (\vec{\omega}\times\vec{r})_i$$

$$= -\omega^2 (A_1)_i$$

$\Rightarrow$ all odd terms

$$(A_{2n+1})_i = (-1)^n \omega^{2n} (A_1)_i = (-1)^n \omega^{2n} (\vec{\omega}\times\vec{r})_i$$

even terms

$$(A_{2n})_i = [B, (A_{2n-1})_i] = [B, (A_{2(n-1)+1})_i]$$

$$= [B, (-1)^{n-1}\omega^{2(n-1)} (A_1)_i]$$

$$= (-1)^{n-1}\omega^{2(n-1)} \underbrace{[B, (A_1)_i]}_{=(A_2)_i}$$

$$= (-1)^{n-1}\omega^{2(n-1)} \left(\vec{\omega}\cdot\vec{r}\ \vec{\omega} - \vec{\omega}^2\vec{r}\right)_i$$

Now everything

$$u^+ r_i u = \sum_{n=0}^{\infty} \frac{1}{n!} A_n \qquad (A_0 = r_i)$$

$$= r_i + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} A_{2n+1} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} A_{2n}$$

$$= r_i + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \omega^{2n+1}}_{= \sin \omega} \left(\frac{\vec{\omega} \times \vec{r}}{\omega} \right)_i$$

$$+ \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \omega^{2n}}_{= \cos \omega - 1} \frac{-1}{\omega^2} \left(\vec{\omega} \cdot \vec{r}\, \vec{\omega} - \vec{\omega}^2 \vec{r} \right)_i$$

$$= r_i + \frac{\sin \omega}{\omega} \left(\vec{\omega} \times \vec{r} \right)_i + (\cos \omega - 1) \frac{1}{\omega^2} \left(\omega^2 \vec{r} - \vec{\omega} \cdot \vec{r}\, \vec{\omega} \right)_i$$

$$= r_i - r_i + \frac{\vec{\omega} \cdot \vec{r}}{\omega^2} \omega_i + \cos \omega \left(r_i - \frac{\vec{\omega} \cdot \vec{r}}{\omega^2} \omega_i \right)$$

$$+ \frac{\sin \omega}{\omega} \left(\vec{\omega} \times \vec{r} \right)_i$$

$$= \left(\frac{\omega_i \omega_j}{\omega^2} + \cos \omega \left(\delta_{ij} - \frac{\omega_i \omega_j}{\omega^2} \right) + \frac{\sin \omega}{\omega} \varepsilon_{ikj} \omega_k \right) r_j$$

$$= R_{ij}(\vec{\omega})\, r_j$$

$\Rightarrow$ Operator $\vec{r}$ transforms like vector remember rotation matrix:

$\rightarrow$ rotate $\vec{r}$ about $\vec{\omega}$

ie axis $\vec{\omega}/\omega$, angle ω.

$\rightarrow$ use $\vec{n} = \frac{\vec{\omega}}{\omega}$; decompose $\vec{r}$:

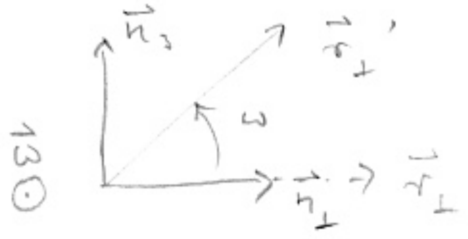
$$\vec{r}_{\parallel} = (\vec{r}\cdot\vec{n})\,\vec{n}$$

$$\vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel} = \vec{r} - (\vec{r}\cdot\vec{n})\,\vec{n}$$

$$(\rightarrow \vec{n}_{\perp} = \vec{r}_{\perp} / r_{\perp})$$

$$\vec{n}_3 = \vec{n} \times \vec{n}_{\perp}$$

$\leadsto$



$$\vec{r}_{\perp}' = (\cos\omega\, \vec{n}_{\perp} + \sin\omega\, \vec{n}_3)\, r_{\perp}$$

$$\vec{r}_{\parallel}' = \vec{r}_{\parallel}$$

hence

$$\vec{r}' = \vec{r}_{\parallel}' + \vec{r}_{\perp}'$$

$$= (\vec{r}\cdot\vec{n})\vec{n} + r_{\perp}\cos\omega\, \vec{n}_{\perp} + r_{\perp}\sin\omega\, \vec{n}_3$$

write in components:

$$(\vec{r}_{\parallel})_i = n_i n_j r_j = \frac{\omega_i \omega_j}{\omega^2} r_j = (\vec{r}_{\parallel}')_i$$

$$(\vec{r}_{\perp}) = r_i - \frac{\omega_i \omega_j}{\omega^2} r_j = \left(\delta_{ij} - \frac{\omega_i \omega_j}{\omega^2}\right) r_j$$

$$(\vec{r}_\perp^{\,2}) = \left(\delta_{ij} - \frac{\omega_i\omega_j}{\omega^2}\right) r_j \left(\delta_{ik} - \frac{\omega_i\omega_k}{\omega^2}\right) r_k$$

$$= r_j r_k \left(\delta_{jk} - \frac{\omega_j\omega_k}{\omega^2} - \frac{\omega_k\omega_j}{\omega^2} + \frac{\omega^2\omega_j\omega_k}{\omega^4}\right)$$

$$= r_j r_k \left(\delta_{jk} - \frac{\omega_j\omega_k}{\omega^2}\right)$$

$$r_\perp \cos\omega \, \vec{n}_\perp = \cos\omega \, \vec{r}_\perp \quad \checkmark$$

$$r_\perp \vec{n}_3 = r_\perp \vec{n} \times \vec{n}_\perp = \vec{n} \times \vec{r}_\perp = \vec{n} \times (\vec{r} - \vec{r}_\parallel)$$

$$= \vec{n} \times \vec{r} = \frac{\vec{\omega} \times \vec{r}}{\omega} = \frac{1}{\omega} \vec{\omega} \times \vec{r}$$

$$(r_\perp \vec{n}_3)_i = \frac{1}{\omega} \varepsilon_{ijk} \omega_j r_k$$

$\rightsquigarrow$ simplified version of $\vec{r}\,'$

$$\vec{r}\,' = \frac{\vec{\omega}\cdot\vec{r}}{\omega^2}\vec{\omega} + \cos\omega \, \vec{r}_\perp + \sin\omega \, \frac{\vec{\omega}\times\vec{r}}{\omega}$$

$$\Leftrightarrow (\vec{r}\,')_i = \left[\frac{\omega_i\omega_j}{\omega^2} + \cos\omega \left(\delta_{ij} - \frac{\omega_i\omega_j}{\omega^2}\right) - \frac{\sin\omega}{\omega} \varepsilon_{ijk}\omega_k \right] r_j$$

$\Rightarrow$ rotation matrix $R(\vec{\omega})_{ij}$

$$\boxed{R(\vec{\omega})_{ij} = \frac{\omega_i\omega_j}{\omega^2} + \cos\omega \left(\delta_{ij} - \frac{\omega_i\omega_j}{\omega^2}\right) - \frac{\sin\omega}{\omega} \varepsilon_{ijk}\omega_k}$$

$\mathcal{U} = R$? in 3D representation

$$\mathcal{U} = e^{-\frac{i}{\hbar}\vec{\omega}\cdot\vec{L}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\vec{\omega}\cdot\vec{L}\right)^n$$

with matrices $L_i = \dots$ we have

$$-\frac{i}{\hbar}\vec{\omega}\cdot\vec{L} = \begin{pmatrix} 0 & -\omega_3 & +\omega_2 \\ +\omega_3 & 0 & +\omega_1 \\ -\omega_2 & +\omega_1 & 0 \end{pmatrix} = \alpha$$

$$\rightsquigarrow \alpha_{ij} = -\varepsilon_{ijk}\,\omega_k$$

$$(\alpha^2)_{ij} = \alpha_{ik}\alpha_{kj} = \varepsilon_{ikl}\,\varepsilon_{kjm}\,\omega_l\omega_m$$

$$= \varepsilon_{lik}\,\varepsilon_{kjm}\,\omega_l\omega_m = (\delta_{lj}\delta_{im} - \delta_{lm}\delta_{ij})\,\omega_l\omega_m$$

$$= \omega_i\omega_j - \omega^2\delta_{ij}$$

$$(\alpha^3)_{ij} = \alpha_{ik}(\alpha^2)_{kj} = (-\varepsilon_{ikl}\,\omega_l)(\omega_k\omega_j - \omega^2\delta_{kj})$$

$$= 0 + \omega^2\varepsilon_{ijl}\,\omega_l = (-\omega^2)(-\varepsilon_{ijl}\,\omega_l)$$

$$= -\omega^2\,\alpha_{ij}$$

$$\Rightarrow (\alpha^{2n})_{ij} = (-\omega^2)^n\left[\delta_{ij} - \frac{\omega_i\omega_j}{\omega^2}\right] = (-1)^n\omega^{2n}[\;]$$

$$(\alpha^{2n+1})_{ij} = (-1)^n(\omega^2)^n\,\omega\left(-\varepsilon_{ijk}\frac{\omega_k}{\omega}\right)$$

$$= (-1)^n\,\omega^{2n+1}\left(-\varepsilon_{ijk}\frac{\omega_k}{\omega}\right)$$

Hence

$$U_{ij} = 1_{ij} + \overbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \omega^{2n}}^{\cos\omega - 1} \left(\delta_{ij} - \frac{\omega_i \omega_j}{\omega^2} \right)$$

$$+ \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \omega^{2n+1}}_{\sin\omega} \left(- \varepsilon_{ijk} \frac{\omega_k}{\omega} \right)$$

$$= \frac{\omega_i \omega_j}{\omega^2} + \cos\omega \left(\delta_{ij} - \frac{\omega_i \omega_j}{\omega^2} \right)$$

$$+ \frac{\sin\omega}{\omega} \left(- \varepsilon_{ijk} \, \omega_k \right)$$

$$= R(\vec{\omega})_{ij} \quad \checkmark \qquad \text{qed}$$

Schwinger Oszillator Model of Angular momentum.

$$J_\pm = \hbar\, a^+_\pm a_\mp \qquad J_z = \frac{\hbar}{2}\left(a_+^+ a_+ - a_-^+ a_-\right)$$

$$N = a_+^+ a_+ + a_-^+ a_-$$

prove that we have angular momentum.

from $[a_\pm, a_\pm^+] = 1$, $[a_+, a_-] = [a_+^+, a_-] = 0$ (independent oszillators).

ie show $[J_z, J_\pm] = \pm\hbar J_\pm$ (a)

$$[J^2, J_z] = 0$$

$$J^2 = \hbar^2 \frac{N}{2}\left(\frac{N}{2} + 1\right) \text{ and } [J_+, J_-]$$

$\rightarrow$ $[J_z, J_+] = \frac{\hbar^2}{2}\Big(a_+^+ a_+ a_+^+ a_- - a_-^+ a_- a_+^+ a_-$

$- a_+^+ a_- a_+^+ a_+ + a_+^+ a_- a_-^+ a_-\Big)$

$= \frac{\hbar^2}{2}\Big(a_+^+ a_+ a_+^+ a_- - a_+^+ a_+ a_+^+ a_- + a_+^+ a_-$

$- a_-^+ a_- a_+^+ a_- + a_-^+ a_- a_+^+ a_- + a_+^+ a_-\Big)$

$= \frac{\hbar^2}{2} 2 a_+^+ a_- = \hbar^2 a_+^+ a_- = +\hbar J_+$

$$\underline{[\partial_z, \partial_-]} = \frac{\hbar^2}{2}\Big(a_+^\dagger a_+ a_-^\dagger a_+ - a_-^\dagger a_- a_-^\dagger a_+$$

$$- a_-^\dagger a_+ a_+^\dagger a_+ + a_-^\dagger a_+ a_-^\dagger a_- \Big)$$

$$= \frac{\hbar^2}{2}\Big(a_+^\dagger a_+ a_-^\dagger a_+ - (a_+^\dagger a_+ a_-^\dagger a_+ + a_-^\dagger a_+)$$

$$- a_-^\dagger a_- a_-^\dagger a_+ + (a_-^\dagger a_- a_-^\dagger a_+ - a_-^\dagger a_+) \Big)$$

$$= \frac{\hbar^2}{2}\left(- a_-^\dagger a_+ - a_-^\dagger a_+ \right) = -\hbar^2 a_-^\dagger a_+ = \underline{-\hbar \partial_-}$$

$$\underline{[\partial_+, \partial_-]} = \hbar^2 \left(a_+^\dagger a_- a_-^\dagger a_+ - a_-^\dagger a_+ a_+^\dagger a_- \right)$$

$$= \hbar^2\Big(\underbrace{a_+^\dagger a_+ a_-^\dagger a_- + a_+^\dagger a_+ - a_-^\dagger a_- a_+^\dagger a_+}_{=0\ \text{(first and third terms)}} - a_-^\dagger a_- \Big)$$

$$= \hbar^2 \left(a_+^\dagger a_+ - a_-^\dagger a_- \right) = \underline{2\hbar\, \partial_z}$$

$$\partial^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = ?$$

consider $\partial_+ \partial_- = (\partial_x + i\partial_y)(\partial_x - i\partial_y)$

$$= \partial_x^2 + \partial_y^2 + i\underbrace{[\partial_y, \partial_x]} = \partial_x^2 + \partial_y^2 + i(-i\hbar\partial_z)$$

$$= \partial_x^2 + \partial_y^2 + \hbar \partial_z$$

$$\Rightarrow \partial^2 = \partial_+\partial_- - \hbar\partial_z + \partial_z^2$$

folgt aus $[\partial_+, \partial_-]$

now $[\partial^2, \partial_z] = [\partial_+\partial_- - \hbar\partial_z + \partial_z^2, \partial_z) = (\partial_+\partial_-, \partial_z]$

$$= \partial_+\partial_-\partial_z - \partial_z\partial_+\partial_- =$$

$$J_+J_-J_z - J_zJ_+J_- =$$

$$= J_+J_-J_z - J_+J_zJ_- - \hbar J_+J_-$$

$$= J_+J_-J_z - J_+J_-J_z + \hbar J_+J_- - \hbar J_+J_- = 0$$

$[J^2, J_z] = 0$ ✓ as for any angular momentum, of course.

then

$$J^2 = J_+J_- - \hbar J_z + J_z^2$$

$$= \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ - \frac{\hbar^2}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$$

$$+ \frac{\hbar^2}{4}\left(a_+^\dagger a_+ a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_-\right)$$

$$J^2 = \frac{\hbar^2}{2} \underbrace{a_+^\dagger a_+ a_-^\dagger a_-}_{= \frac{1}{4} + \frac{1}{4}} + \frac{\hbar^2}{2} a_+^\dagger a_+ + \frac{\hbar^2}{2} a_-^\dagger a_-$$

$$+ \frac{\hbar^2}{4}\left(a_+^\dagger a_+ a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_-\right)$$

$$= \frac{\hbar^2}{4}\Big(a_+^\dagger a_+ \left(a_+^\dagger a_+ + a_-^\dagger a_- + 2\right)$$

$$+ a_-^\dagger a_- \left(a_+^\dagger a_+ + a_-^\dagger a_- + 2\right) \Big)$$

$$= \frac{\hbar^2}{4}\left(a_+^\dagger a_+ + a_-^\dagger a_-\right)\left(a_+^\dagger a_+ + a_-^\dagger a_- + 2\right)$$

$$= \hbar^2 \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

(used $N = a_+^\dagger a_+ + a_-^\dagger a_-$).