# Moderne Theoretische Physik I
# Grundlagen der Quantenmechanik

**Summer Semester 2024**
**Exercise Sheet 1**

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**The problems whose solutions you need to upload are designated with stars.**

## ⋆ Problem 1 ⋆ Dirac Delta Distribution

We consider the family of functions $\delta_\sigma(x) = \alpha e^{-x^2/\sigma^2}$ and want to show that in the limit $\sigma \to 0$, it corresponds to the Dirac distribution $\delta(x)$. The latter is not a function in the strict sense but rather a probability distribution. It is defined such that the probability density is zero when $x \neq 0$. However, it is still normalized so that the integral over the real axis equals 1. This means that the probability distribution is non-zero at $x = 0$.

1. Determine $\alpha$ so that each function $\delta_\sigma(x)$ is normalized according to
$$\int_{-\infty}^{\infty} \mathrm{d}x \delta_\sigma(x) = 1. \tag{1}$$
2. Show that for each fixed $x$, the limit $\lim_{\sigma \to 0} \delta_\sigma(x) = \delta(x)$ is satisfied.
3. Determine, using the $\sigma \to 0$ limiting behavior ($f$ is a smooth function)
$$\int_{-\infty}^{\infty} \mathrm{d}x \delta_\sigma(x) f(x) \to \int_{-\infty}^{\infty} \mathrm{d}x \delta(x) f(x) = f(0), \tag{2}$$
what is the analogous limiting behavior of $\int_{-\infty}^{\infty} \mathrm{d}x \delta'_\sigma(x) f(x)$.
4. Prove the relationship
$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta\left(f(x)\right) = \sum_i \frac{1}{|f'(x_i)|} \tag{3}$$
where $x_i$ are the simple zeros of the function $f(x)$. (Simple zero means that the function's derivative is finite there. Assume $f$ is smooth and has only simple zeros.)

## ⋆ Problem 2 ⋆ Expectation values of a Gaussian wave function

Consider the wave function ($\sigma > 0$):
$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \tag{4}$$

1. Show that the given wave function is normalized: $\int_{-\infty}^{\infty} |\psi(x)|^2 = 1$.

2. Calculate the expectation value of $x$:

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \mathrm{d}x |\psi(x)|^2 x. \tag{5}$$

This is the first moment of the distribution, also called the mean.

3. Calculate the expectation value of $x^2$:

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} \mathrm{d}x |\psi(x)|^2 x^2. \tag{6}$$

Using these two quantities, we can calculate the second moment of the distribution, $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. This quantity is also called the standard deviation.

4. Calculate the expectation value of the momentum $\hat{p}$:

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \frac{\hbar}{i} \partial_x \psi(x). \tag{7}$$

5. Calculate the expectation value of $\hat{p}^2$ and standard deviation of the $\hat{p}$:

$$\langle \hat{p}^2 \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \left(\frac{\hbar}{i}\right)^2 \partial_x^2 \psi(x), \tag{8}$$

$$\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} \tag{9}$$

6. Calculate the product $\Delta x \cdot \Delta p$. Compare it with Heisenberg's uncertainty relation $\Delta x \cdot \Delta p \geq \hbar/2$.

## Problem 3 Spectral density in a box

Consider an electromagnetic field in a cubic box with volume $V = L^3$. A simple estimate for the number of free electromagnetic modes can be obtained by requiring periodic boundary conditions on the vector potential ($\omega_{\mathbf{k}} = c|\mathbf{k}|$)

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} \tag{10}$$

1. Show that this condition leads to a quantization of the k-states and determine this. Specifically, show that $k = \frac{2\pi}{L}(n_x, n_y, n_z)$ with $n_x, n_y, n_z \in \mathbb{Z}$.

2. Use the quantization condition from 1 to derive an expression for the number of modes $dN$ in the interval $[k, k + dk]$; here $k = |\mathbf{k}|$. Keep in mind that the vector potential is transverse to the k-vector, i.e., $\mathbf{A} \cdot \mathbf{k} = 0$.

3. Calculate the spectral energy density $u(\omega)$ ($u(\omega)d\omega$ is the energy per volume in the interval $[\omega, \omega + d\omega]$) in thermal equilibrium. To do this, use the classic equipartition principle, which states that each mode contributes the energy $k_B T$. Explain why this assumption is problematic.

4. Planck's law of radiation

$$u(\omega) = \frac{\eta \omega^3}{\pi^2 c^3} \frac{1}{e^{\eta\omega/k_B T} - 1} \tag{11}$$

avoids the problem mentioned above. Determine the behavior of this radiation law at small and large frequencies and compare with the results of part 3. Specify the units of $\eta$ and interpret the quantity $\eta\omega$.