Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 3

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Particle in a box

Consider a particle of mass m that lies in the infinitely deep well

$$V(x) = \begin{cases} 0, & \text{for } 0 < x < L, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1}$$

1. If $x(t=0) = x_0 \in \langle 0, L \rangle$ and $p(t=0) = p_0 \neq 0$, what is the solution of the classical equations of motion? By imposing the Sommerfeld quantization condition

$$\oint \mathrm{d}x\, p = \int_0^T \mathrm{d}t\, \dot{x}(t) p(t) = n \cdot 2\pi\hbar, \tag{2}$$

   where the integral goes over one period of motion (T is the time period) and $n \in \mathbb{N}$, find what p_0 must equal. Calculate the corresponding energy.

2. If the wavefunction of the particle at $t = 0$ equals

$$\psi(x) = a_1 \sin(n_1 \pi x/L) + a_2 \sin(n_2 \pi x/L), \tag{3}$$

   where $a_1, a_2 \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{N}$ with $a_1 \neq 0, a_2 \neq 0, n_1 \neq n_2$, then what is the wavefunction at later times?

3. Now using this wavefunction, calculate the corresponding expectation values of $\hat{x}$ and $\hat{p} = -\mathrm{i}\hbar\partial_x$ at later times. For concreteness, set $n_1 = 1$ and $n_2 = 2$.

4. According to Ehrenfest's theorem:

$$\partial_t \langle \hat{x} \rangle = \frac{\langle \hat{p} \rangle}{m}, \qquad \partial_t \langle \hat{p} \rangle = -\langle V'(\hat{x}) \rangle. \tag{4}$$

   Do the results of parts 1 and 3 agree with this? Explain any apparent disagreements.

⋆ Problem 2 ⋆ Delta potential well

Consider the delta potential well

$$V(x) = -V_0\, \delta(x), \tag{5}$$

where $V_0 > 0$.

1. By integrating the stationary Schrödinger equation from $x = -\epsilon$ to ϵ for small $\epsilon > 0$, derive the condition that the wavefunction must obey at $x = 0$. You may assume that ψ is continuous at $x = 0$.

2. Solve the stationary Schrödinger equation for bounded states, that is, find all eigenfunctions of the Hamiltonian which have a finite norm.

3. What are the corresponding energies? Explicitly calculate them by evaluating the integral arising in the Hamiltonian average $\left\langle \hat{H} \right\rangle$. Evaluate the kinetic energy in two ways: as $\langle \psi | \hat{p}^2 \psi \rangle$ and as $\langle \hat{p}\psi | \hat{p}\psi \rangle$.

Problem 3 Linear differential equations as Schrödinger equations

1. Consider the wave equation describing the vibrations of a string:

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t), \tag{6}$$

where the speed of sound has been set to unity. Derive the effective Hamiltonian $\hat{H}$ (which is a 2×2 matrix here) arising in

$$\mathrm{i}\sigma_z \partial_t \psi = \hat{H}\psi \tag{7}$$

which describes the evolution of the multicomponent wavefunction

$$\psi = \begin{pmatrix} u + \mathrm{i}\partial_t u \\ u - \mathrm{i}\partial_t u \end{pmatrix}. \tag{8}$$

Here

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

2. Physically, $u^*(x,t) = u(x,t)$ must be real. What is the analogous reality condition on $\psi(x,t)$? Show that Eq. (7) preserves it.

3. Next, consider Maxwell's equations (with $\epsilon_0 = \mu_0 = 1$)

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{10}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j} + \partial_t \boldsymbol{E}. \tag{11}$$

Introduce the complex three-component field

$$\boldsymbol{\psi} = \boldsymbol{B} - \mathrm{i}\boldsymbol{E}. \tag{12}$$

Find the effective Hamiltonian (which is a 3×3 operator matrix) and the right-hand side in

$$(\mathrm{i}\partial_t - \hat{H})\boldsymbol{\psi} = ? \tag{13}$$

Replace all spatial derivatives with the momentum operators $\hat{p}_x = -\mathrm{i}\partial_x$, $\hat{p}_y = -\mathrm{i}\partial_y$, and $\hat{p}_z = -\mathrm{i}\partial_z$ in $\hat{H}$.

4. Confirm that $\hat{H}$ is Hermitian with respect to the scalar product

$$\langle \boldsymbol{\psi} | \boldsymbol{\phi} \rangle = \int \mathrm{d}^3 r \, \boldsymbol{\psi}^*(\boldsymbol{r}) \cdot \boldsymbol{\phi}(\boldsymbol{r}). \tag{14}$$

What boundary conditions must $\boldsymbol{\psi}, \boldsymbol{\phi}$ satisfy?