Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 4

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The problems whose solutions you need to upload are designated with stars.

$\star$ Problem 1 $\star$ $\frac{1}{2}$-spin of electrons

Consider the following three operators:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

There operators are $\frac{1}{2}$-spin operators.

1. Show that the above spin operators satisfying following commutation relation

$$[\hat{S}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k \tag{2}$$

where ϵ_{ijk} is a Levi-Civita symbol:

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (i,j,k) = (x,y,z),(y,z,x) \text{ and } (z,x,y) \\ -1, & \text{if } (i,j,k) = (z,y,x),(x,z,y) \text{ and } (y,x,z) \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } i = k. \end{cases} \tag{3}$$

2. Show that eigenvalues of the spin operators are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ and find the corresponding eigenvectors of all three spin operators. We will denote the eigenvectors of the spin operator $\hat{S}_i$ with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ as $|i :\uparrow\rangle$ and $|i :\downarrow\rangle$ respectively. Finally show that the state $|x :\uparrow\rangle$ is superposition of $|z :\uparrow\rangle$ and $|z :\downarrow\rangle$ with equal probabilities.

3. Consider an operator defined as follows:

$$\hat{S}_{\hat{n}} = \vec{S} \cdot \hat{n} \tag{4}$$

where $\hat{\vec{S}} = \hat{S}_x\hat{x} + \hat{S}_y\hat{y} + \hat{S}_z\hat{z}$ and $\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$. ($\hat{x}$, $\hat{y}$ and $\hat{z}$ are not operators! They are unit vectors.)

Show that eigenvalues of $S_{\hat{n}}$ are same to the $\hat{S}_i$ (i.e. $\pm\frac{\hbar}{2}$). If we denote the eigenvectors with $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ as $|\hat{n} :\uparrow\rangle$ and $|\hat{n} :\downarrow\rangle$, show that $\langle \hat{n} :\uparrow |\vec{S}|\hat{n} :\uparrow\rangle = \frac{\hbar}{2}\hat{n}$ and $\langle \hat{n} :\downarrow |\vec{S}|\hat{n} :\downarrow\rangle = -\frac{\hbar}{2}\hat{n}$.

4. Show that

$$e^{i\theta\frac{\hat{S}_{\hat{n}}}{\hbar}} = \hat{1}\cos\frac{\theta}{2} + i\frac{2}{\hbar}S_{\hat{n}}\sin\frac{\theta}{2} \tag{5}$$

where $\hat{1}$ is a 2×2 identity matrix. (Hint: see Problem 1 in Exercise Sheet 2)

5. Using the above results, show that

$$U(\theta,\phi)\hat{S}_z U^\dagger(\theta,\phi) = \hat{S}_{\hat{n}} \tag{6}$$

where $U(\theta,\phi) = e^{-i\phi\frac{\hat{S}_z}{\hbar}} e^{-i\theta\frac{\hat{S}_y}{\hbar}}$. Here $U(\theta,\phi)$ is the rotation matrix with Euler angles θ and ϕ.

⋆ Problem 2 ⋆ Projectors and spectral decomposition

Let $\hat{A}$ be a Hermitian operator with a discrete, non-degenerate spectrum (every eigenvalue has only one eigenvector)

$$\hat{A}|n\rangle = a_n|n\rangle,\ n \in \mathrm{N} \tag{7}$$

where a_n is the eigenvalue of the (normalized) eigenstate $|n\rangle$. We define

$$\hat{P}_n = |n\rangle\langle n| \tag{8}$$

as a projector on the eigenspace spanned by the eigenvector $|n\rangle$.

1. Show that for any state $|\psi\rangle$ a following equation holds

$$\hat{A}\hat{P}_n|\psi\rangle = a_n\hat{P}_n|\psi\rangle \tag{9}$$

2. Show that this is true

$$\hat{P}_n\hat{P}_m = \delta_{nm}\hat{P}_n \tag{10}$$

3. Express $\hat{A}$ using the operators $\hat{P}_n$. To do this, use the completeness of the $|n\rangle$ states (i.e. $\sum_n |n\rangle\langle n| = \hat{1}$)
4. The probability for a state $|\psi\rangle$ to be measured with eigenvalue n is given by

$$P_{|\psi\rangle}(n) = |\langle n|\psi\rangle|^2 \tag{11}$$

Express $P_{|\psi\rangle}(n)$ using $\hat{P}_n$ and $|\psi\rangle$.

Now let $\hat{B}$ be a Hermitian operator with a purely continuous spectrum (such as the momentum operator $\hat{p}$):

$$\hat{B}|b\rangle = b|b\rangle,\ b \in \mathrm{R} \tag{12}$$

These eigenstates satisfy following properties:

$$\langle b|b'\rangle = \delta(b-b'),\ \int db|b\rangle\langle b| = \hat{1} \tag{13}$$

In analogy to Eq. (8), we now define the projector for the eigenvalues between α and β ($\alpha < \beta$)

$$\hat{P}_{[\alpha,\beta]} = \int_\alpha^\beta db|b\rangle\langle b|. \tag{14}$$

5. Show that $\hat{P}_{[\alpha,\beta]}\hat{P}_{[\gamma,\delta]} = \hat{P}_{[\alpha,\beta]\cap[\gamma,\delta]}$.
6. Express the probability of measuring a value within the interval $[\alpha,\beta]$ when measuring the observable $\hat{B}$ from the state $|\psi\rangle$ using the projector $\hat{P}_{[\alpha,\beta]}$.
7. Now let $\hat{B} = \hat{p}$ be the momentum operator. Let the wave function in momentum representation be

$$\psi(p) = \langle p|\psi\rangle = \begin{cases} \frac{1}{\sqrt{2p_0}}, & \text{for} |p| < p_0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Using the above projectors, calculate the probability of measuring a particle with momentum $p > 0$ given a measurement of $|\psi\rangle$.

Problem 3 Free particle in a homogeneous field

A homogeneous field acting on a particle is determined by the potential

$$V(x) = -Fx \tag{16}$$

The Schrödinger equation for this system has the form of the Airy differential equation in spatial space

$$f^{''}(x) - xf(x) = 0 \tag{17}$$

However, the solution to this problem is easier to find in momentum space.

1. Give the momentum space representation for the Schrödinger equation of a particle in a homogeneous field.
2. Determine the wave function $\psi(p)$ that solves the Schrödinger equation,
3. Show that this solution, when transformed back into real space, becomes the implicit equation for the Airy function

$$A_i(\xi) = \int \frac{du}{\pi} \cos\left(\frac{u^3}{3} + \xi\right). \tag{18}$$