Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 5

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Hermite's polynomials

The Hamiltonian of the simple harmonic oscillator (SHO) in the x-basis and its energy eigenvalues are given by

$$\Big(-\frac{\hbar^2}{2m}\partial_x^2+\frac{m\omega^2}{2}x^2\Big)\psi_n(x)=E_n\psi_n(x),\ E_n=\hbar\omega\Big(n+\frac{1}{2}\Big) \tag{1}$$

The eigenfunctions $\psi_n(x)$ are closely related to Hermite's polynomials

$$H_n(z)=(-1)^n e^{z^2}\partial_z^n e^{-z^2},\ \ n\geq 0 \tag{2}$$

1. First, show that the function e^{-t^2+2zt} is a generating function of Hermite polynomials, i.e.

$$e^{-t^2+2zt}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(z) \tag{3}$$

   (Hint: use the Taylor expansion of $e^{-(z-t)^2}$)

2. Using the above result, derive the following recursion relations for H_n:

$$\partial_z H_n(z)=2nH_{n-1}(z),\ \ n\geq 1 \tag{4}$$

   and

$$H_{n+1}(z)=2zH_n(z)-2nH_{n-1}(z),\ \ n\geq 1 \tag{5}$$

   Derive the following differential equation using Eqs. (4) and (5)

$$[\partial_z^2-2z\partial_z+2n]H_n(z)=0 \tag{6}$$

   (Hint: Eqs. (4) and (5) can be proven by differentiating Eq. (3) with respect to z or with respect to t)

3. Show the orthogonality of the Hermite polynomials,

$$\int_{-\infty}^{\infty}dz e^{-z^2}H_n(z)H_m(z)=0,\ \text{for } n\neq m \tag{7}$$

   (Hint: manipulate Eq. (6) and integrate it over z)

⋆ Problem 2 ⋆ Two-dimensional harmonic oscillator

We consider the two-dimensional harmonic oscillator with the Hamilton operator

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{m\omega^2}{2}(\hat{x}_1^2 + \hat{x}_2^2) \tag{8}$$

where $\hat{x}_i$ and $\hat{p}_i$ satisfy the commutation relations: $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$ and $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ where i, j=1,2.

1. Based on Heisenberg's uncertainty relation, derive a lower bound of the ground state energy.
2. From the position and momentum operators $\hat{x}_i$, $\hat{p}_j$, we define creation and annihilation operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ as follows:

$$\hat{a}_i = \alpha\hat{x}_i + i\beta\hat{p}_i, \tag{9}$$

$$\hat{a}_i^\dagger = \alpha\hat{x}_i - i\beta\hat{p}_j \tag{10}$$

   where α and β are real numbers.

   Determine α and β so that:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \tag{11}$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \tag{12}$$

$$\hat{H} = \sum_{j=1}^{2} \hbar\omega\left(\hat{N}_j + \frac{1}{2}\right) \tag{13}$$

   where $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$.

3. Prove the following identities:

$$[\hat{N}_i, \hat{a}_j] = -\hat{a}_j\delta_{ij}, \tag{14}$$

$$[\hat{N}_i, \hat{a}_j^\dagger] = \hat{a}_j^\dagger\delta_{ij}, \tag{15}$$

$$[\hat{N}_i, \hat{N}_j] = 0 \tag{16}$$

4. Because $[\hat{N}_1, \hat{N}_2] = 0$ we can find common eigenstates for $\hat{N}_1$ and $\hat{N}_2$

$$\hat{N}_1|n_1, n_2\rangle = n_1|n_1, n_2\rangle, \tag{17}$$

$$\hat{N}_2|n_1, n_2\rangle = n_2|n_1, n_2\rangle \tag{18}$$

   Calculate the effect of $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$ on the state $|n_1, n_2\rangle$. To do this, calculate the eigenvalues of $\hat{N}_1$ and $\hat{N}_2$ from the respective states $\hat{a}_1|n_1, n_2\rangle$, $\hat{a}_2|n_1, n_2\rangle$, $\hat{a}_1^\dagger|n_1, n_2\rangle$, $\hat{a}_2^\dagger|n_1, n_2\rangle$.

5. Now what are the eigenstates and eigenenergies of the two-dimensional harmonic oscillator? Why does $n_1, n_2 \in N_0$ have to apply? (Hint: you can use the result of part 1 of this assignment that the energy eigenvalues are bounded from below)

Problem 3 Cauchy-Schwarz inequality

1. Derive the Cauchy-Schwarz inequality

$$|\mathbf{v}|^2|\mathbf{u}|^2 \geq |\mathbf{v}\cdot\mathbf{u}|^2 \tag{19}$$

   by using the fact that

$$(\mathbf{v} - \lambda\mathbf{u})^2 \geq 0 \tag{20}$$

   and minimizing $(\mathbf{v} - \lambda\mathbf{u})^2$ with respect to λ. Here $\mathbf{v}$ and $\mathbf{u}$ are real-valued vectors and λ is a real number.

2. Can we extend the above result to the complex-valued vector case?

$$(\mathbf{v}^*\cdot\mathbf{v})|(\mathbf{u}^*\cdot\mathbf{u})| \geq |\mathbf{v}^*\cdot\mathbf{u}|^2 \tag{21}$$