Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 6

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Double Dirac delta potential

Consider a particle of mass m subject to the potential

$$V(x) = -V_0\,\delta(x-L) - V_0\,\delta(x+L), \tag{1}$$

where $V_0 > 0$.

1. What condition must a wavefunction $\psi(x)$ that satisfies the stationary Schrödinger equation obey at $x = \pm L$?
2. Find all the bound states of this potential. No need to normalize the states.
3. Explicitly find the energies, assuming they are small. What does this mean for LV_0? Discuss.

⋆ Problem 2 ⋆ Coherent states

Consider the destruction operator

$$\hat{a} = \frac{\hat{x} + \mathrm{i}\,\hat{p}}{\sqrt{2}} \tag{2}$$

with all units set to unity ($m = \omega_0 = \hbar = 1$).

1. Find the eigenstates $\phi_z(x)$ of the destruction operator in real space (x basis). That is, solve $\hat{a}\phi_z(x) = z\phi_z(x)$ for $z \in \mathbb{C}$. Normalize the states according to $\langle\phi_z|\phi_z\rangle = \mathrm{e}^{|z|^2}$ and chose their global phase so that they depend only on z, and not on $\operatorname{Re} z$ or $\operatorname{Im} z$ separately. These states are called coherent states.
2. Given a wavefunction in real space $\psi(x) \equiv \langle x|\psi\rangle$, find the corresponding wavefunction $(\mathcal{B}\psi)(z) \equiv \langle\phi_z|\psi\rangle$ in the coherent state basis $\phi_z(x) = \langle x|\phi_z\rangle$. This change of basis is known as a Bargmann transformation (cf. Fourier transformation).
3. Find how the operators $\hat{x}$, $\hat{p}$, $\hat{a}$, and $\hat{a}^\dagger$ act in the coherent state basis.
4. Verify that $[\hat{x},\hat{p}] = \mathrm{i}$ and $[\hat{a},\hat{a}^\dagger] = 1$ still holds in the coherent-state-basis representation.

Problem 3 Heisenberg's uncertainty principle

Consider a normalized wavefunction $\psi(x)$.

1. If $\langle\hat{x}\rangle = x_0$, what modification of $\psi(x)$ has $\langle\hat{x}\rangle = \langle\psi|\hat{x}|\psi\rangle = 0$?
2. If $\langle\hat{p}\rangle = p_0$, what modification of $\psi(x)$ has $\langle\hat{p}\rangle = \langle\psi|\hat{p}|\psi\rangle = 0$? Here $\hat{p} = -\mathrm{i}\hbar\partial_x$, as usual.

Hence, without loss of generality, we shall now consider a $\psi(x)$ with $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$.

3. Prove Heisenberg's uncertainty principle by applying the Cauchy-Schwarz inequality to the states $|\phi\rangle \equiv \hat{x}\,|\psi\rangle$ and $|\chi\rangle \equiv \hat{p}\,|\psi\rangle$.
4. Now recall when the Cauchy-Schwarz inequality is an equality. Exploit this fact to derive the states which minimize $\sigma_x\sigma_p$. Explicitly check this by evaluating the standard deviations $\sigma_{x,p}$.