Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

Summer Semester 2024
Exercise Sheet 6

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Due date: 07. 06. 2024.

The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Double Dirac delta potential

Consider a particle of mass m subject to the potential

$$V(x) = -V_0\,\delta(x-L) - V_0\,\delta(x+L), \tag{1}$$

where $V_0 > 0$.

1. What condition must a wavefunction $\psi(x)$ that satisfies the stationary Schrödinger equation obey at $x = \pm L$?
2. Find all the bound states of this potential. No need to normalize the states.
3. Explicitly find the energies, assuming they are small. What does this mean for LV_0? Discuss.

Solution 1

1. As in Exercise Sheet 3, we integrate the stationary Schrödinger equation

$$\frac{-\hbar^2}{2m}\partial_x^2\psi(x) - V_0\,\delta(x-L)\psi(x) - V_0\,\delta(x+L)\psi(x) = E\psi(x)$$

around $x = \pm L$ to get

$$\frac{-\hbar^2}{2m}\left[\psi'(-L+\epsilon) - \psi'(-L-\epsilon)\right] = V_0\psi(-L),$$
$$\frac{-\hbar^2}{2m}\left[\psi'(L+\epsilon) - \psi'(L-\epsilon)\right] = V_0\psi(L)$$

for infinitesimal $\epsilon > 0$.

2. For $x \neq \pm L$, the solutions of the stationary Schrödinger equation have the form of exponentials:

$$\psi(x) = \begin{cases} A\mathrm{e}^{\kappa(x+L)}, & \text{for } x < -L, \\ B\cosh(\kappa x) + C\sinh(\kappa x), & \text{for } -L < x < L, \\ D\mathrm{e}^{-\kappa(x-L)}, & \text{for } L < x, \end{cases}$$

where the solutions which exponentially diverge at infinity have been dropped (they aren't localized/bounded). Here

$$\kappa = \frac{\sqrt{-2mE}}{\hbar}.$$

The wavefunction must be continuous at $x = \pm L$. Hence

$$A = B\cosh(\kappa L) - C\sinh(\kappa L),$$
$$D = B\cosh(\kappa L) + C\sinh(\kappa L),$$

The $x = \pm L$ conditions in addition give:

$$(\lambda - \kappa)A = \kappa[B\sinh(\kappa L) - C\cosh(\kappa L)],$$
$$(\lambda - \kappa)D = \kappa[B\sinh(\kappa L) + C\cosh(\kappa L)],$$

where

$$\lambda = \frac{2mV_0}{\hbar^2}.$$

By dividing the two sets of equations

$$\lambda - \kappa = \kappa\frac{B\sinh(\kappa L) - C\cosh(\kappa L)}{B\cosh(\kappa L) - C\sinh(\kappa L)} = \kappa\frac{B\sinh(\kappa L) + C\cosh(\kappa L)}{B\cosh(\kappa L) + C\sinh(\kappa L)}$$

which implies that

$$BC = 0.$$

So either $C = 0$, in which case we have an even solution, or $B = 0$, in which case we have an odd solution.

The even-parity solution has $A = B\cosh(\kappa L) = D$ and its energy is found by inverting the transcedental equation

$$(1 + \tanh\kappa_* L)\kappa_* L = \lambda L \qquad \Longrightarrow \; E = -\frac{\hbar^2}{2m}\kappa_*^2.$$

The odd-parity solution has $-A = C\sinh(\kappa L) = D$ and its energy is found by inverting

$$(1 + \coth\kappa_* L)\kappa_* L = \lambda L \qquad \Longrightarrow \; E = -\frac{\hbar^2}{2m}\kappa_*^2.$$

3. For small $\kappa_* L$, we find that

$$(1 + \tanh\kappa_* L)\kappa_* L = \kappa_* L + \cdots$$
$$(1 + \coth\kappa_* L)\kappa_* L = 1 + \kappa_* L + \cdots$$

Hence the even-parity solution exists for all λ, and in the limit of small λ it has the energy $E = -\frac{\hbar^2}{2m}\lambda^2 = -2mV_0^2/\hbar^2$. This agrees with Exercise Sheet 3, Problem 2.

The odd-parity solution only exists for $\lambda L \geq 1$. It has the energy $E = -\frac{\hbar^2}{2mL^2}(\lambda L - 1)^2$.

⋆ Problem 2 ⋆ Coherent states

Consider the destruction operator

$$\hat{a} = \frac{\hat{x} + \mathrm{i}\,\hat{p}}{\sqrt{2}} \tag{2}$$

with all units set to unity ($m = \omega_0 = \hbar = 1$).

1. Find the eigenstates $\phi_z(x)$ of the destruction operator in real space (x basis). That is, solve $\hat{a}\phi_z(x) = z\phi_z(x)$ for $z \in \mathbb{C}$. Normalize the states according to $\langle \phi_z | \phi_z \rangle = \mathrm{e}^{|z|^2}$ and chose their global phase so that they depend only on z, and not on $\operatorname{Re} z$ or $\operatorname{Im} z$ separately. These states are called coherent states.
2. Given a wavefunction in real space $\psi(x) \equiv \langle x|\psi\rangle$, find the corresponding wavefunction $(\mathcal{B}\psi)(z) \equiv \langle \phi_z|\psi\rangle$ in the coherent state basis $\phi_z(x) = \langle x|\phi_z\rangle$. This change of basis is known as a Bargmann transformation (cf. Fourier transformation).
3. Find how the operators $\hat{x}$, $\hat{p}$, $\hat{a}$, and $\hat{a}^\dagger$ act in the coherent state basis.
4. Verify that $[\hat{x}, \hat{p}] = \mathrm{i}$ and $[\hat{a}, \hat{a}^\dagger] = 1$ still holds in the coherent-state-basis representation.

Solution 2

1. The normalized-to-unity solution of

$$\hat{a}\phi_z(x) = z\phi_z(x),$$
$$(\partial_x + x)\phi_z(x) = \sqrt{2}z\phi_z(x)$$

is

$$\tilde{\phi}_z(x) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2} + \sqrt{2}xz - |\operatorname{Re} z|^2\right).$$

By multiplying with $\mathrm{e}^{|z|^2/2}$ we get $\langle \phi_z|\phi_z\rangle = \mathrm{e}^{|z|^2}$, and by multiplying with $\mathrm{e}^{-\mathrm{i}\operatorname{Im} z \operatorname{Re} z}$ we get

$$\phi_z(x) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2} + \sqrt{2}xz - \frac{z^2}{2}\right).$$

2. The Bargmann transformation is found by inserting a resolution of unity in the x basis:

$$\begin{aligned}
(\mathcal{B}\psi)(z) \equiv \langle \phi_z|\psi\rangle = \langle \phi_z|\hat{1}|\psi\rangle = \langle \phi_z| \cdot \int \mathrm{d}x\, |x\rangle\langle x| \cdot |\psi\rangle \\
= \int \mathrm{d}x\, \langle \phi_z|x\rangle \langle x|\psi\rangle = \int \mathrm{d}x\, \langle x|\phi_z\rangle^* \psi(x) \\
= \frac{1}{\pi^{1/4}} \int \mathrm{d}x \exp\left(-\frac{x^2}{2} + \sqrt{2}xz^* - \frac{z^{*2}}{2}\right)\psi(x).
\end{aligned}$$

3. Since

$$\begin{aligned}
\partial_x\phi_z(x) &= (\sqrt{2}z - x)\phi_z(x), \\
\partial_z\phi_z(x) &= (\sqrt{2}x - z)\phi_z(x), \\
\implies x\phi_z(x) &= \frac{z + \partial_z}{\sqrt{2}}\phi_z(x), \\
\implies \partial_x\phi_z(x) &= \frac{z - \partial_z}{\sqrt{2}}\phi_z(x), \\
\hat{a}\phi_z(x) &= z\phi_z(x), \\
\implies \hat{a}^\dagger\phi_z(x) &= \frac{\hat{x} - \mathrm{i}\,\hat{p}}{\sqrt{2}}\phi_z(x) = \frac{x - \partial_x}{\sqrt{2}}\phi_z(x) = \partial_z\phi_z(x)
\end{aligned}$$

and $\phi_z^*(x) = \phi_{z^*}(x)$, it follows that

$$\begin{aligned}
\hat{x}(\mathcal{B}\psi)(z) &= \langle \phi_z|\hat{x}\psi\rangle = \langle \hat{x}\phi_z|\psi\rangle = \frac{z^* + \partial_{z^*}}{\sqrt{2}}(\mathcal{B}\psi)(z), \\
\hat{p}(\mathcal{B}\psi)(z) &= \langle \phi_z|\hat{p}\psi\rangle = \langle \hat{p}\phi_z|\psi\rangle = \mathrm{i}\frac{z^* - \partial_{z^*}}{\sqrt{2}}(\mathcal{B}\psi)(z), \\
\hat{a}^\dagger(\mathcal{B}\psi)(z) &= \langle \phi_z|\hat{a}^\dagger\psi\rangle = \langle \hat{a}\phi_z|\psi\rangle = z^*(\mathcal{B}\psi)(z), \\
\hat{a}(\mathcal{B}\psi)(z) &= \langle \phi_z|\hat{a}\psi\rangle = \langle \hat{a}^\dagger\phi_z|\psi\rangle = \partial_{z^*}(\mathcal{B}\psi)(z).
\end{aligned}$$

4. This immediately follows from the fact that $[\partial_{z^*}, z^*] = 1$.

Problem 3 Heisenberg's uncertainty principle

Consider a normalized wavefunction $\psi(x)$.

1. If $\langle\hat{x}\rangle = x_0$, what modification of $\psi(x)$ has $\langle\hat{x}\rangle = \langle\psi|\hat{x}|\psi\rangle = 0$?
2. If $\langle\hat{p}\rangle = p_0$, what modification of $\psi(x)$ has $\langle\hat{p}\rangle = \langle\psi|\hat{p}|\psi\rangle = 0$? Here $\hat{p} = -\mathrm{i}\hbar\partial_x$, as usual.

Hence, without loss of generality, we shall now consider a $\psi(x)$ with $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$.

3. Prove Heisenberg's uncertainty principle by applying the Cauchy-Schwarz inequality to the states $|\phi\rangle \equiv \hat{x}\,|\psi\rangle$ and $|\chi\rangle \equiv \hat{p}\,|\psi\rangle$.
4. Now recall when the Cauchy-Schwarz inequality is an equality. Exploit this fact to derive the states which minimize $\sigma_x\sigma_p$. Explicitly check this by evaluating the standard deviations $\sigma_{x,p}$.

Solution 3

1. One simply translates the wavefunction: $\phi(x) \equiv \psi(x_0 + x)$ has $\langle\phi|\hat{x}|\phi\rangle = 0$.
2. One multiplies the wavefunction by a phase factor: $\phi(x) \equiv \mathrm{e}^{-\mathrm{i}p_0x/\hbar}\psi(x)$ has $\langle\phi|\hat{p}|\phi\rangle = 0$. This is the same as a translation in momentum space.
3. The Cauchy-Schwarz inequality states that

$$\begin{aligned} &|\langle\phi|\phi\rangle| \cdot |\langle\chi|\chi\rangle| \geq |\langle\phi|\chi\rangle|^2 \\ \implies &|\langle\hat{x}\psi|\hat{x}\psi\rangle| \cdot |\langle\hat{p}\psi|\hat{p}\psi\rangle| \geq |\langle\hat{x}\psi|\hat{p}\psi\rangle|^2 \\ \implies &\left|\langle\psi|\hat{x}^2|\psi\rangle\right| \cdot \left|\langle\psi|\hat{p}^2|\psi\rangle\right| \geq |\langle\psi|\hat{x}\hat{p}|\psi\rangle|^2 \\ \implies &\sigma_x^2\sigma_p^2 \geq |\langle\psi|\hat{x}\hat{p}|\psi\rangle|^2 \end{aligned}$$

   Now we use the fact that in $\hat{x}\hat{p} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{1}{2}(\hat{x}\hat{p} - \hat{p}\hat{x})$, the anticommutator is Hermitian, whereas the commutator is anti-Hermitian. Thus in

$$z = \langle\psi|\hat{x}\hat{p}|\psi\rangle = \left\langle\psi\middle|\tfrac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\middle|\psi\right\rangle + \left\langle\psi\middle|\tfrac{1}{2}(\hat{x}\hat{p} - \hat{p}\hat{x})\middle|\psi\right\rangle = \operatorname{Re} z + \mathrm{i}\operatorname{Im} z$$

   the anticommutator gives the real part, whereas the commutator gives the imaginary part. Since we know that $|z|^2 \geq |\operatorname{Im} z|^2$, it follows that

$$\sigma_x^2\sigma_p^2 \geq \frac{1}{4}\,|\langle\psi|[\hat{x},\hat{p}]|\psi\rangle|^2 = \hbar^2/4 \implies \sigma_x\sigma_p \geq \hbar/2.$$

4. The Cauchy-Schwarz inequality is saturated when the two vectors are proportional (parallel) to each other. Thus we need to solve

$$\begin{aligned} \hat{p}\,|\psi\rangle &= |\chi\rangle = \tilde{\lambda}\,|\phi\rangle = \tilde{\lambda}\hat{x}\,|\psi\rangle \\ -\mathrm{i}\hbar\partial_x\psi(x) &= (2\mathrm{i}\hbar\lambda)\cdot x\psi(x) \end{aligned}$$

   for $\lambda = \tilde{\lambda}/(2\mathrm{i}\hbar) \in \mathbb{C}$. The solution is

$$\psi(x) = A\exp\bigl(-\lambda x^2\bigr).$$

For $\operatorname{Re}\lambda > 0$, this is normalizeable, with $A = \sqrt[4]{2\operatorname{Re}\lambda_1/\pi}$. Evaluating the averages, one finds that

$$\begin{aligned}
\langle\psi|\hat{x}^2|\psi\rangle &= \frac{1}{4\operatorname{Re}\lambda} = \sigma_x^2, \\
\langle\psi|\hat{p}^2|\psi\rangle &= \frac{(\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}{\operatorname{Re}\lambda} = \sigma_p^2, \\
\langle\psi|\hat{x}\hat{p}|\psi\rangle &= -\frac{\operatorname{Re}\lambda + \mathrm{i}\operatorname{Im}\lambda}{2\operatorname{Re}\lambda}, \\
\langle\psi|\hat{p}\hat{x}|\psi\rangle &= \frac{\operatorname{Re}\lambda - \mathrm{i}\operatorname{Im}\lambda}{2\operatorname{Re}\lambda}.
\end{aligned}$$

So even though $\sigma_x\sigma_p = |\langle\psi|\hat{x}\hat{p}|\psi\rangle|$ holds for all λ, Heisenberg's inequality only holds when $\operatorname{Im}\lambda = 0$. Indeed, this is expected because in Heisenberg's inequality we dropped the anticommutator part, which is equivalent to demanding that $\langle\psi|\hat{x}\hat{p}|\psi\rangle = -\langle\psi|\hat{p}\hat{x}|\psi\rangle \implies \operatorname{Im}\lambda = 0$. Thus Gaussian wavepackets of the general form

$$\psi(x) = \frac{1}{\sqrt[4]{2\sigma_x^2\pi}} \exp\left(\mathrm{i}\frac{p_0 x}{\hbar} - \frac{(x-x_0)^2}{4\sigma_x^2}\right).$$

minimize Heisenberg's inequality.