Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 8

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Operators as transformation generators

The exponential of an operator $\hat{A}$ is defined as

$$\exp(\hat{A}) = \mathrm{e}^{\hat{A}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^n. \tag{1}$$

(Recall also the discussion of Problem 1 of Exercise Sheet 2.)

1. Consider a Hermitian operator $\hat{Q}$ and a state $|\psi\rangle$. If $|\psi'\rangle = \mathrm{e}^{-\mathrm{i}\hat{Q}} |\psi\rangle$, then find the formal expression for the $\hat{A}'$ entering
$$\left\langle \psi' \middle| \hat{A} \middle| \psi' \right\rangle = \left\langle \psi \middle| \hat{A}' \middle| \psi \right\rangle. \tag{2}$$

2. Now define the operator
$$\hat{A}_t = \mathrm{e}^{\mathrm{i}t\hat{Q}} \hat{A} \mathrm{e}^{-\mathrm{i}t\hat{Q}}, \tag{3}$$
where t is a real parameter. Find its n-th derivative at $t = 0$:
$$\left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} \hat{A}_t \right|_{t=0} = ? \tag{4}$$

3. Prove the Baker-Campbell-Hausdorff lemma
$$\mathrm{e}^{\mathrm{i}t\hat{Q}} \hat{A} \mathrm{e}^{-\mathrm{i}t\hat{Q}} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} \operatorname{ad}_{\hat{Q}}^n \hat{A}. \tag{5}$$
Here $\operatorname{ad}_{\hat{Q}} \hat{A} \equiv [\hat{Q}, \hat{A}]$ so that $\operatorname{ad}_{\hat{Q}}^2 \hat{A} = [\hat{Q}, [\hat{Q}, \hat{A}]]$, $\operatorname{ad}_{\hat{Q}}^3 \hat{A} = [\hat{Q}, [\hat{Q}, [\hat{Q}, \hat{A}]]]$, etc.

4. Calculate
$$\mathrm{e}^{\mathrm{i}a\hat{p}/\hbar} \hat{x} \mathrm{e}^{-\mathrm{i}a\hat{p}/\hbar} = ? \tag{6}$$
$$\mathrm{e}^{-\mathrm{i}k\hat{x}/\hbar} \hat{p} \mathrm{e}^{\mathrm{i}k\hat{x}/\hbar} = ? \tag{7}$$
where $\hat{x}$ and $\hat{p}_x$ are the position and momentum operators and a, k are real numbers. Given a state $|\psi\rangle$, what does $\mathrm{e}^{-\mathrm{i}a\hat{p}/\hbar} |\psi\rangle$ and $\mathrm{e}^{\mathrm{i}k\hat{x}/\hbar} |\psi\rangle$ physically do to this state?

5. Now consider the angular momentum along z, $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$. Calculate

$$e^{i\varphi \hat{L}_z/\hbar} \hat{x} e^{-i\varphi \hat{L}_z/\hbar} = ? \tag{8}$$

$$e^{i\varphi \hat{L}_z/\hbar} \hat{y} e^{-i\varphi \hat{L}_z/\hbar} = ? \tag{9}$$

   What sort of transformation does this represent physically?

⋆ Problem 2 ⋆ Particle in a central potential in two dimensions

Consider a particle subject to a radially symmetric potential $V(r)$, $r = \sqrt{x^2 + y^2}$, in two dimensions.

1. Formulate the Schrödinger equation in polar coordinates. Using separation of variables, derive the radial Schrödinger equation for a state of fixed energy E.
2. Now use a substitution $R(r) = r^{-\alpha} u(r)$ and choose α so that one of the terms vanishes. Find the effective potential $V_{\text{eff}}(r)$ from the resulting radial Schrödinger equation.
3. Next, find the energies when $V(r) = -e^2/r$. You may invoke the results from the analysis of the 3D hydrogen atom from the lectures, if relevant.

Problem 3 Particle in a modified Coulomb potential

Consider a particle in three dimensions subject to the spherically symmetric potential:

$$V(r) = -\frac{Ze^2}{r} + \frac{\gamma}{r^2}. \tag{10}$$

Find the energies of the bound states for this modified Coulomb potential. What happens when $\gamma < 0$? (Hint: follow the steps you went through when solving the hydrogen atom problem during class.)