Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 10

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Fun with orbital angular momentum

The orbital angular momentum operator is given by $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z) = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. In spherical coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \ \text{ with } r = \sqrt{x^2+y^2+z^2} \tag{1}$$

and the gradient is given by

$$\nabla_{r,\theta,\phi} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\theta \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{2}$$

with

$$\hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}, \tag{3}$$

$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z, \tag{4}$$

$$\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y \tag{5}$$

1. Show that the angular momentum operator in spherical coordinates has the following form:

$$\hat{L}_x = \frac{\hbar}{i}\Big(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big),\ \hat{L}_y = \frac{\hbar}{i}\Big(\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big),\ \hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \tag{6}$$

2. Suppose a particle described by the following wave function

$$\psi(\mathbf{r}) = (x+y+2z)Ne^{-r^2/\alpha^2} \tag{7}$$

where N and α both are real numbers and N is a normalization constant.
By applying

$$\hat{\mathbf{L}}^2 = -\hbar^2\Big(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta}\Big) \tag{8}$$

to the state $\psi(\mathbf{r})$, show that $\psi(\mathbf{r})$ is an eigenstate of the $\hat{\mathbf{L}}^2$. i.e.

$$\hat{\mathbf{L}}^2\psi(\mathbf{r}) = l(l+1)\hbar^2\psi(\mathbf{r}) \tag{9}$$

and determine the value of l.

3. Now express the wave function Eq. (7) by a superposition of suitable spherical harmonics. Which values can be measured for the z-component $\hat{L}_z$ of the orbital angular momentum? With what probability are these measured?

⋆ Problem 2 ⋆ Particles in a magnetic field - Landau levels

Consider a particle of charge q in a homogeneous magnetic field $\mathbf{B} = B\hat{e}_z$. A clever choice of the vector potential $\mathbf{A}$ in this case is given by the Landau gauge with $\mathbf{A} = -By\hat{e}_x$. Assuming the particle is restricted to the $x-y$ plane (as in a two-dimensional electron gas), the Hamiltonian is

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}\right)^2 = \frac{1}{2m}\left(\left(\hat{p}_x + \frac{q}{c}B\hat{y}\right)^2 + \hat{p}_y^2\right) \tag{10}$$

1. Show $[\hat{H}, \hat{p}_x] = 0$, and use the knowledge of the eigenfunctions of $\hat{p}_x$ to make a separation of variable approach for the wave function $\psi(x, y)$.
2. Show that the Schrödinger equation can be brought to the form of a one-dimensional harmonic oscillator and find the characteristic frequency ω_c of the eigenenergies $E_n = \hbar\omega_c(n + \frac{1}{2})$ where $n \geq 0$.
3. Now specify the corresponding eigenfunctions $\psi_{n,p_x}(x, y)$. Use the magnetic length scale $l_B = \sqrt{\frac{\hbar c}{qB}}$ in expressing the $\psi_{n,p_x}(x, y)$.

Apparently the eigenfunctions depend on the quantum number p_x, but the energies do not. Thus the Landau energy levels are strongly degenerate. This degeneracy plays an important role for physical applications (e.g., de Haas-van Alphen effect). We now want to determine these degeneracies for a sample with an area $A = L_x L_y$.

4 Determine the quantization of $\hat{p}_x$, assuming periodic boundary conditions $\psi(x+L_x, y) = \psi(x, y)$. Also find the distance between adjacent values of p_{x,n_x}. i.e. $\Delta p_x = p_{x,n_x+1} - p_{x,nx}$.
(Hint: To do this, perform a discrete Fourier transformation $\phi(x) = \frac{1}{\sqrt{L_x}} \sum_{p_x} e^{-ip_x x/\hbar}\phi_{p_x}$.)

5 A restriction on the permitted values of p_x can be found by the condition that the position of the potential minimum $y_0 = \frac{cp_x}{qB}$ must lie within the dimensions of the sample, i.e. $0 < y_0 < L_y$. From this, determine the I_{p_x} which is length of the interval of permitted p_x values, and the number $N = \frac{I_{p_x}}{\Delta p_x}$ (= degree of degeneracy of each Landau level).

Problem 3 Detection of directional quantization in the magnetic field and spin precession

In a groundbreaking experiment done by Stern and Gerlach, they were able to demonstrate the directional quantization of the angular momentum. They used a setup with a strongly in-homogeneous magnetic field and observed that a silver atom beam is split into two beams. This setup, also called Stern-Gerlach apparatus shown in Fig. 1, can also be used to investigate spin precession in more detail.

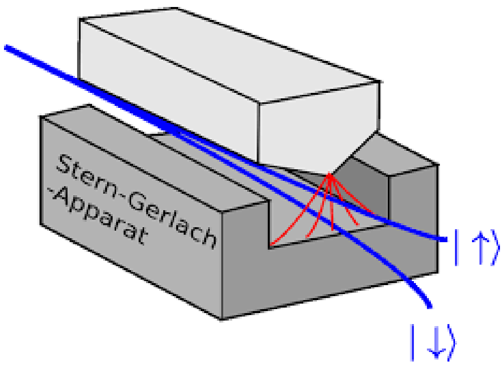

Figure 1: Stern-Gerlach apparatus

Here we shall consider two Stern-Gerlach apparatuses arranged one behind the other. The first has an inhomogeneous magnetic field along the z-direction which splits the electron beam into $|\uparrow\rangle$ and $|\downarrow\rangle$ states. The second apparatus has an inhomogeneous magnetic field along the x-direction which also splits the electron beam, this time into spins along $\pm\hat{x}$. A homogeneous magnetic field B_y is applied between the apparatuses in the y-direction and leads to a precision of the spin during the flight time T between the two Stern-Gerlach apparatuses. Two points are now observed on a detector screen behind the second apparatus.

1. Sketch the experimental setup including the beam path.
2. The intensities of the two observed points depend on the magnetic field B_y and the time of flight T. Calculate this dependence for the two possible prepared initial states ($|\uparrow\rangle$ and $|\downarrow\rangle$).