Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 10

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Fun with orbital angular momentum

The orbital angular momentum operator is given by $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z) = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. In spherical coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \ \text{ with } r = \sqrt{x^2+y^2+z^2} \tag{1}$$

and the gradient is given by

$$\nabla_{r,\theta,\phi} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\theta \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{2}$$

with

$$\hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}, \tag{3}$$
$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z, \tag{4}$$
$$\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y \tag{5}$$

1. Show that the angular momentum operator in spherical coordinates has the following form:
$$\hat{L}_x = \frac{\hbar}{i}\Big(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big),\ \hat{L}_y = \frac{\hbar}{i}\Big(\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big),\ \hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \tag{6}$$

2. Suppose a particle described by the following wave function
$$\psi(\mathbf{r}) = (x+y+2z)Ne^{-r^2/\alpha^2} \tag{7}$$
where N and α both are real numbers and N is a normalization constant.
By applying
$$\hat{\mathbf{L}}^2 = -\hbar^2\Big(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta}\Big) \tag{8}$$
to the state $\psi(\mathbf{r})$, show that $\psi(\mathbf{r})$ is an eigenstate of the $\hat{\mathbf{L}}^2$. i.e.
$$\hat{\mathbf{L}}^2\psi(\mathbf{r}) = l(l+1)\hbar^2\psi(\mathbf{r}) \tag{9}$$
and determine the value of l.

3. Now express the wave function Eq. (7) by a superposition of suitable spherical harmonics. Which values can be measured for the z-component $\hat{L}_z$ of the orbital angular momentum? With what probability are these measured?

Solution 1

1. We now show that the angular momentum operator in spherical coordinates has the form:

$$\hat{L}_x = \frac{\hbar}{i}\Big(- \sin\phi \frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big),\ \hat{L}_y = \frac{\hbar}{i}\Big(\cos\phi \frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big),\ \hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \tag{10}$$

   Using Eq. (2) and the fact that

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi,\ \hat{e}_\theta \times \hat{e}_\phi = \hat{e}_r,\ \hat{e}_\phi \times \hat{e}_r = \hat{e}_\theta, \tag{11}$$

   the angular momentum operator is now given by

$$\begin{aligned}\frac{i}{\hbar}\hat{\mathbf{L}} &= \hat{\mathbf{r}} \times \hat{\mathbf{p}} = \hat{\mathbf{r}} \times \nabla = r\hat{e}_r \times \Big(\hat{e}_r\partial_r + \hat{e}_\theta\frac{1}{r}\partial_\theta + \hat{e}_\phi\frac{1}{r\sin\theta}\partial_\phi\Big) \\ &= \hat{e}_\phi\partial_\theta - \hat{e}_\theta\frac{1}{\sin\theta}\partial_\phi \\ &= \Big(- \sin\phi - \frac{\cos\phi}{\tan\theta}\partial_\phi\Big)\hat{e}_x + \Big(\cos\phi\partial_\theta - \frac{\sin\phi}{\tan\theta}\partial_\phi\Big)\hat{e}_y + \partial_\phi\hat{e}_z\end{aligned} \tag{12}$$

2. First we write the given wave function in spherical coordinates

$$\begin{aligned}\psi(r,\theta,\phi) &= (r\sin\theta\cos\phi + r\sin\theta\sin\phi + 2r\cos\theta)Ne^{-r^2/\alpha^2} \\ &= (\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta)f(r)\end{aligned} \tag{13}$$

   where the $f(r) = rNe^{-r^2/\alpha^2}$

   Now we apply $\hat{\mathbf{L}}^2$ to the wave function

$$\hat{\mathbf{L}}^2\psi(r,\theta,\phi) = -\hbar^2\Big(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta}\Big)(\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta)f(r) \tag{14}$$

$$\frac{\partial^2}{\partial\theta^2}(\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta) = -(\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta), \tag{15}$$

$$\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta) = -\frac{1}{\sin\theta}(\cos\phi + \sin\phi), \tag{16}$$

$$\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta) = \frac{\cos^2\theta}{\sin\theta}(\cos\phi + \sin\phi) - 2\cos\theta \tag{17}$$

   As a result,

$$\hat{\mathbf{L}}^2\psi(r,\theta,\phi) = 2\hbar^2(\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta)f(r) = 2\hbar\psi(r,\theta,\phi) \tag{18}$$

   Thus $\psi(r,\theta,\phi)$ is an eigenfunction with the eigenvalue $2\hbar = l(l+1)\hbar$, from which $l = 1$ can be read directly.

3. The spherical harmonics are complete and orthogonal. The coefficients could therefore be found by projecting onto these states. However, since we already know that $l = 1$, the state can only be expressed by a superposition of the functions $Y_{l=1}^m(\theta,\phi)$. We therefore only need the three spherical harmonics

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta,\ \ Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi},\ \ Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \tag{19}$$

   and find the coefficients by comparing with Eq. (13).

$$\begin{aligned}\psi(r,\theta,\phi) &= \Big(- \sqrt{\frac{2\pi}{3}}(Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)) + i\sqrt{\frac{2\pi}{3}}(Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)) + 2\sqrt{\frac{4\pi}{3}}Y_1^0(\theta,\phi)\Big)f(r) \\ &= \Big(c_1Y_1^1(\theta,\phi) + c_0Y_1^0(\theta,\phi) + c_{-1}Y_1^{-1}(\theta,\phi)\Big)f(r)\end{aligned} \tag{20}$$

where

$$c_1 = -\sqrt{\frac{2\pi}{3}}(1-i),\ c_0 = 4\sqrt{\frac{\pi}{3}},\ c_{-1} = \sqrt{\frac{3\pi}{3}}(1+i). \tag{21}$$

In a measurement of $\hat{L}_z$, the values $\hbar m$ with $m = -1, 0, 1$ can be found. We have not normalized the wave function, i.e. $f(r)$ is only known up to a constant factor N. However, when calculating the measurement probabilities of the various angular momentum values, $f(r)$ is eliminated

$$P(m=-1) = \frac{|c_{-1}|^2}{|c_{-1}|^2 + |c_0|^2 + |c_1|^2} = \frac{1}{6} \tag{22}$$

$$P(m=0) = \frac{|c_0|^2}{|c_{-1}|^2 + |c_0|^2 + |c_1|^2} = \frac{2}{3} \tag{23}$$

$$P(m=1) = \frac{|c_1|^2}{|c_{-1}|^2 + |c_0|^2 + |c_1|^2} = \frac{1}{6} \tag{24}$$

⋆ Problem 2 ⋆ Particles in a magnetic field - Landau levels

Consider a particle of charge q in a homogeneous magnetic field $\mathbf{B} = B\hat{e}_z$. A clever choice of the vector potential $\mathbf{A}$ in this case is given by the Landau gauge with $\mathbf{A} = -By\hat{e}_x$. Assuming the particle is restricted to the $x-y$ plane (as in a two-dimensional electron gas), the Hamiltonian is

$$\hat{H} = \frac{1}{2m}\Big(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}\Big)^2 = \frac{1}{2m}\left(\Big(\hat{p}_x + \frac{q}{c}B\hat{y}\Big)^2 + \hat{p}_y^2\right) \tag{25}$$

1. Show $[\hat{H}, \hat{p}_x] = 0$, and use the knowledge of the eigenfunctions of $\hat{p}_x$ to make a separation of variable approach for the wave function $\psi(x,y)$.
2. Show that the Schrödinger equation can be brought to the form of a one-dimensional harmonic oscillator and find the characteristic frequency ω_c of the eigenenergies $E_n = \hbar\omega_c(n + \frac{1}{2})$ where $n \geq 0$.
3. Now specify the corresponding eigenfunctions $\psi_{n,p_x}(x,y)$. Use the magnetic length scale $l_B = \sqrt{\frac{\hbar c}{qB}}$ in expressing the $\psi_{n,p_x}(x,y)$.

Apparently the eigenfunctions depend on the quantum number p_x, but the energies do not. Thus the Landau energy levels are strongly degenerate. This degeneracy plays an important role for physical applications (e.g., de Haas-van Alphen effect). We now want to determine these degeneracies for a sample with an area $A = L_x L_y$.

4 Determine the quantization of $\hat{p}_x$, assuming periodic boundary conditions $\psi(x+L_x, y) = \psi(x,y)$. Also find the distance between adjacent values of p_{x,n_x}. i.e. $\Delta p_x = p_{x,n_x+1} - p_{x,nx}$.
(Hint: To do this, perform a discrete Fourier transformation $\phi(x) = \frac{1}{\sqrt{L_x}}\sum_{p_x} e^{-ip_x x/\hbar}\phi_{p_x}$.)

5 A restriction on the permitted values of p_x can be found by the condition that the position of the potential minimum $y_0 = \frac{cp_x}{qB}$ must lie within the dimensions of the sample, i.e. $0 < y_0 < L_y$. From this, determine the I_{p_x} which is length of the interval of permitted p_x values, and the number $N = \frac{I_{p_x}}{\Delta p_x}$ (= degree of degeneracy of each Landau level).

Solution 2

1. We can easily show that $[\hat{H}, \hat{p}_x] = 0$ from the fact that $[\hat{y}, \hat{p}_x] = 0$. So we can make an Ansatz: $\psi(x,y) = e^{ip_x x/\hbar}\chi(y)$.

2. Inserting the ansatz into Schrödinger equation, we immediately get

$$\Big[\frac{\hat{p}_y^2}{2m}+\frac{1}{2}m\Big(\frac{qB}{mc}\Big)^2\Big(y+\frac{cp_x}{qB}\Big)^2\Big]\chi(y)=E\chi(y) \tag{26}$$

Let us now introduce the characteristic frequency $\omega_c=\frac{qB}{mc}$, and the displacement $y_0=\frac{cp_x}{qB}$ and also a variable shift by $\tilde{y}=y+y_0$, $\hat{p}_y=\hat{p}_{\tilde{y}}$. Then we get

$$\Big[\frac{\hat{p}_{\tilde{y}}^2}{2m}+\frac{1}{2}m\omega_c^2\tilde{y}^2\Big]\chi(\tilde{y}-y_0)=E\chi(\tilde{y}-y_0) \tag{27}$$

Here we have a harmonic oscillator with the energies $E_n=\hbar\omega_c(n+\frac{1}{2})$, and the eigenfunctions of the harmonic oscillator

$$\chi(\tilde{y}-y_0)=\tilde{\chi}(\tilde{y})=\Big(\frac{m\omega_c}{\pi\hbar}\Big)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\Big(\sqrt{\frac{m\omega_c}{\hbar}}\tilde{y}\Big)e^{-\frac{1}{2}\frac{m\omega_c}{\hbar}\tilde{y}^2} \tag{28}$$

3. It is $\chi(y)=\tilde{\chi}(y+y_0)$. With the magnetic length scale $l_B=\sqrt{\frac{\hbar}{m\omega_c}}=\sqrt{\frac{\hbar c}{qB}}$, we can write the eigenfunctions as follows

$$\begin{aligned}\psi_{n,p_x}(x,y)&=e^{ip_xx/\hbar}\chi(y)=e^{ip_xx/\hbar}\tilde{\chi}(y+y_0)\\&=e^{ip_xx/\hbar}\frac{1}{\pi^{1/4}\sqrt{l_B}\sqrt{2^n n!}}H_n\Big(\frac{y+y_0}{l_B}\Big)e^{-\frac{(y+y_0)^2}{2l_B^2}}.\end{aligned} \tag{29}$$

4. To determine the quantization under periodic boundary conditions we perform the Fourier transformation

$$\psi(x,y)=\frac{1}{\sqrt{L_x}}\sum_{p_x}e^{-ip_xx/\hbar}\psi_{p_x}(y)=\psi(x+L_x,y)=\frac{1}{\sqrt{L_x}}\sum_{p_x}e^{-ip_x(x+L_x)/\hbar}\psi_{p_x}(y) \tag{30}$$

$$\Rightarrow p_xL_x=2\pi\hbar n_x\ (n_x=0,\pm1,\pm2,\cdots) \tag{31}$$

$$\therefore p_x=\frac{2\pi\hbar}{L_x}n_x. \tag{32}$$

Therefore $\Delta p_x=p_{x,n_x+1}-p_{x,n_x}=\frac{2\pi\hbar}{L_x}$.

5. Now we require that the position of the potential minimum y_0 lies within the sample i.e. $0<y_0<L_y$ and thus obtain the restriction for p_x, $0<p_x<\frac{qBL_y}{c}$. Then the length of the interval of the allowed p_x is $I_{p_x}=\frac{qBL_y}{c}$. As a result, the number of degenerated states is given by

$$N=\frac{I_{p_x}}{\Delta p_x}=\frac{qBA}{2\pi\hbar c} \tag{33}$$

Problem 3 Detection of directional quantization in the magnetic field and spin precession

In a groundbreaking experiment done by Stern and Gerlach, they were able to demonstrate the directional quantization of the angular momentum. They used a setup with a strongly in-homogeneous magnetic field and observed that a silver atom beam is split into two beams. This setup, also called Stern-Gerlach apparatus shown in Fig. 1, can also be used to investigate spin precession in more detail.

Here we shall consider two Stern-Gerlach apparatuses arranged one behind the other. The first has an inhomogeneous magnetic field along the z-direction which splits the electron beam into $|\uparrow\rangle$ and $|\downarrow\rangle$ states. The second apparatus has an inhomogeneous magnetic field along the x-direction which also splits the electron beam, this time into spins along $\pm\hat{x}$. A homogeneous magnetic field B_y is applied between the apparatuses in the y-direction and leads to a precision of the spin during the flight time T between the two Stern-Gerlach apparatuses. Two points are now observed on a detector screen behind the second apparatus.

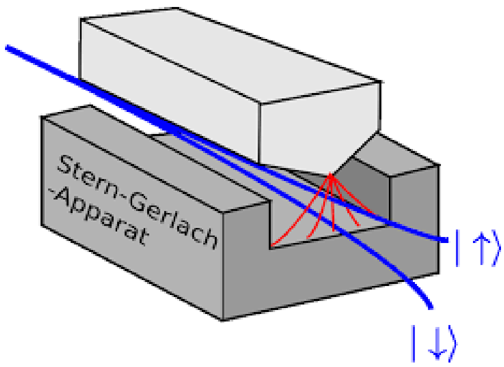
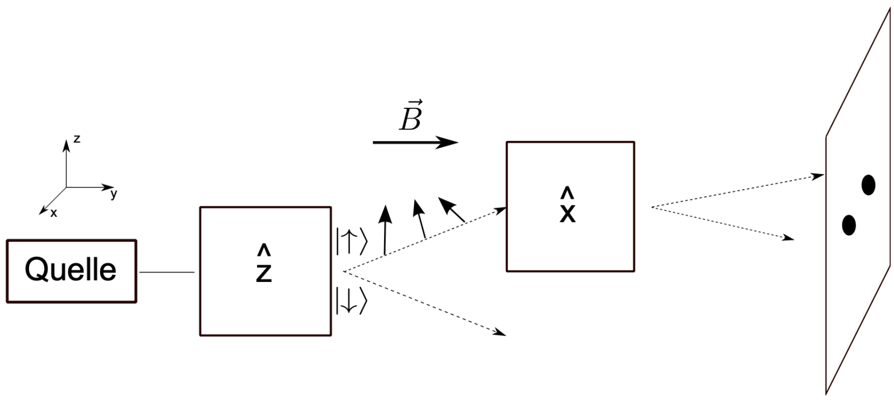


Figure 1: Stern-Gerlach apparatus

1. Sketch the experimental setup including the beam path.
2. The intensities of the two observed points depend on the magnetic field B_y and the time of flight T. Calculate this dependence for the two possible prepared initial states ($|\uparrow\rangle$ and $|\downarrow\rangle$).

Solution 3

1. Sketch



2. According to the first Stern-Gerlach apparatus, the state of the particles is either $|\uparrow\rangle$ (for the upper beam, for example) and $|\downarrow\rangle$ (for the lower beam).

   During the flight time from one device to the other, the states evolve due to the external magnetic field applied in the y-direction

$$|\psi_\uparrow(t)\rangle = \frac{e^{-i\omega t/2}}{2}(|\uparrow\rangle + i|\downarrow\rangle) + \frac{e^{i\omega t/2}}{2}(|\uparrow\rangle - i|\downarrow\rangle) = \cos\frac{\omega t}{2}|\uparrow\rangle - \sin\frac{\omega t}{2}|\downarrow\rangle, \tag{34}$$

$$|\psi_\downarrow(t)\rangle = \cos\frac{\omega t}{2}|\downarrow\rangle + \sin\frac{\omega t}{2}|\uparrow\rangle, \tag{35}$$

$$\tag{36}$$

with $\omega = \frac{\mu_B B}{\hbar}$. The second Stern-Gerlach apparatus splits the states again into the states

$$|\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \tag{37}$$

$$|\downarrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle). \tag{38}$$

Accordingly, the conditional probabilities to measure these after the flight time T are given by

$$P(\uparrow_x, \uparrow) = \left| \frac{1}{\sqrt{2}} \left(\langle \uparrow | + \langle \downarrow | \right) \left(\cos \frac{\omega T}{2} | \uparrow \rangle - \sin \frac{\omega T}{2} | \downarrow \rangle \right) \right|^2 = \frac{1}{2}(1 - \sin \omega T), \tag{39}$$

$$P(\downarrow_x, \uparrow) = \frac{1}{2}(1 + \sin \omega T), \tag{40}$$

$$P(\uparrow_x, \downarrow) = \frac{1}{2}(1 + \sin \omega T), \tag{41}$$

$$P(\downarrow_x, \downarrow) = \frac{1}{2}(1 - \sin \omega T). \tag{42}$$

The intensities of the points on the detector screen are directly proportional to these probabilities.