Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 11

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Entanglement entropy of the 2-site Quantum Ising model in a transverse field

Consider the Hamiltonian for two spin-1/2 particles:

$$\hat{H} = -J\hat{\sigma}_1^z\hat{\sigma}_2^z - h\hat{\sigma}_1^x - h\hat{\sigma}_2^x \quad (J, h \geq 0) \tag{1}$$

where $\hat{\sigma}^x$ and $\hat{\sigma}^z$ are Pauli matrices.

A convenient orthonormal basis of states which spans the full Hilbert space for this model consists of a direct product of eigenstates of $\hat{\sigma}^z$ denoted, for example,

$$|\phi_1\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2, \quad |\phi_2\rangle = |\downarrow\rangle_1 \otimes |\downarrow\rangle_2, \quad |\phi_3\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2, \quad |\phi_4\rangle = |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \tag{2}$$

where $\hat{\sigma}_{1,2}^z|\uparrow\rangle_{1,2} = |\uparrow\rangle_{1,2}$ and $\hat{\sigma}_{1,2}^z|\downarrow\rangle_{1,2} = -|\downarrow\rangle_{1,2}$. **Note:** you may use Wolfram Mathematica for this problem.

1. Find the matrix elements for this Hamiltonian, $h_{\alpha\beta} = \langle\phi_\alpha|\hat{H}|\phi_\beta\rangle$, with $\alpha, \beta = 1, 2, 3, 4$.

2. Find the eigenstate and eigenvalue of the matrix h with the lowest eigenvalue. If one denotes this (normalized) eigenstate as $|\Psi\rangle$, you should be now able to express it as $|\Psi\rangle = \sum_{\alpha=1}^4 A_\alpha|\phi_\alpha\rangle$ with known values of the coefficients A_α.

3. We are interested in the entanglement entropy of the state $|\Psi\rangle$ for a bipartition that divides sites 1 and 2. First consider a density matrix of the full system simply given by $|\Psi\rangle$,

$$\hat{\rho} = |\Psi\rangle\langle\Psi| \tag{3}$$

   To calculate the entanglement entropy between the 2-sites we first need to find the reduced density matrix, which we denote by $\hat{\rho}_1$, for this bipartition. Therefore, calculate

$$\hat{\rho}_1 = Tr_2(\hat{\rho}) \tag{4}$$

   where Tr_2 denotes a trace over the Hilbert space of site 2, that is over the 2-states $|\uparrow\rangle_2, |\downarrow\rangle_2$. This gives the reduced density matrix $\hat{\rho}_1$ for the subsystem consisting of site 1.

4. Re-express the density-matrix operator, $\hat{\rho}_1$ as a 2×2 matrix in the basis $|\uparrow\rangle_1, |\downarrow\rangle_1$, with matrix elements, for example, $\langle\uparrow_1|\hat{\rho}_1|\uparrow\rangle_1$ and so on.

5. Diagonalize your 2× 2 matrix representation of $\hat{\rho}_1$ to obtain its eigenvalues λ_i.
6. The von Neumann (bi-partite entanglement) entropy is defined as,

$$S_1^{vN} = -Tr_1[\hat{\rho}_1 \ln \hat{\rho}_1] = -\sum_i \lambda_i \ln \lambda_i \tag{5}$$

   Calculate S_1^{vN} as a function of h/J and make a sketch of S_1^{vN} versus h/J.

7. What is the value of S_1^{vN} as $\frac{h}{J} \to \infty$? As $\frac{h}{J} \to 0$? Explain the physics behind these two limits.

⋆ Problem 2 ⋆ Operator in Heisenberg picture

Consider a harmonic oscillator with the Hamiltonian

$$H = \hbar\omega\Big(\hat{a}^\dagger \hat{a} + \frac{1}{2}\Big) \tag{6}$$

Where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators respectively. In the Heisenberg picture, in contrast to the Schrödinger picture, the states are time-independent, but the operators for them are time-dependent.

1. Use the Heisenberg equation of motion to calculate the time evolution of the operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.
2. Calculate the time evolution of the operators $\hat{x}(t)$ and $\hat{p}(t)$.
3. Use the results of the previous parts of the task to calculate the commutators $[\hat{x}(t_1), \hat{x}(t_2)]$ and $[\hat{x}(t_1), \hat{p}(t_2)]$ of the now time-dependent operators.
4. At time $t = 0$, the system is in the ground state $|\psi(0)\rangle = |0\rangle$. Calculate the correlation function $\langle \hat{x}(0)\hat{x}(t)\rangle$.

Problem 3 Spin in a time-dependent magnetic field

In the lecture, the Larmor precession of the electron spin in a static magnetic field in the z-direction $\mathbf{B} = B\hat{z}$ was discussed. In the following, the behavior of the spin-$\frac{1}{2}$ particle will be investigated when a periodic magnetic field is additionally applied in the xy-plane.

Consider an electron spin in a time-dependent external magnetic field:

$$\mathbf{B}(t) = B_1 \cos\omega t \hat{e}_x + B_1 \sin\omega t \hat{e}_y + B_0 \hat{e}_z \tag{7}$$

1. Write the Hamiltonian of the system explicitly in matrix form.
2. Show that the problem can be transformed into the problem of an electron in a static magnetic field $\bar{\mathbf{B}} = B_x\hat{e}_x + B_z\hat{e}_y$ by using a sutiable transformation:

$$\psi_\uparrow(t) = a(t)\bar{\psi}_\uparrow(t), \quad \psi_\downarrow(t) = b(t)\bar{\psi}_\downarrow(t). \tag{8}$$

3. Use the time evolution operator $\hat{U}(t, 0)$ to calculate the time evolution of the state $|\bar{\psi}(t)\rangle$ of the electron spin in the effective static magnetic field $\bar{\mathbf{B}}$.
4. At time $t = 0$ the particle is in the state $|\psi(0)\rangle = |\uparrow\rangle$. Calculate the probability $P_\downarrow(t)$ of finding the spin at time t in the state $|\downarrow\rangle$. For which frequency ω of the external field and after which time does the probability become maximum?