Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

Summer Semester 2024
Exercise Sheet 11

Prof. Jörg Schmalian
Iksu Jang, Grgur Palle
Karlsruher Institut für Technologie (KIT)
Due date: 12. 07. 2024.

The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Entanglement entropy of the 2-site Quantum Ising model in a transverse field

Consider the Hamiltonian for two spin-1/2 particles:

$$\hat{H} = -J\hat{\sigma}_1^z\hat{\sigma}_2^z - h\hat{\sigma}_1^x - h\hat{\sigma}_2^x \;\; (J, h \geq 0) \tag{1}$$

where $\hat{\sigma}^x$ and $\hat{\sigma}^z$ are Pauli matrices.

A convenient orthonormal basis of states which spans the full Hilbert space for this model consists of a direct product of eigenstates of $\hat{\sigma}^z$ denoted, for example,

$$|\phi_1\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2, \;\; |\phi_2\rangle = |\downarrow\rangle_1 \otimes |\downarrow\rangle_2, \;\; |\phi_3\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2, \;\; |\phi_4\rangle = |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \tag{2}$$

where $\hat{\sigma}_{1,2}^z|\uparrow\rangle_{1,2} = |\uparrow\rangle_{1,2}$ and $\hat{\sigma}_{1,2}^z|\downarrow\rangle_{1,2} = -|\downarrow\rangle_{1,2}$. **Note:** you may use Wolfram Mathematica for this problem.

1. Find the matrix elements for this Hamiltonian, $h_{\alpha\beta} = \langle\phi_\alpha|\hat{H}|\phi_\beta\rangle$, with $\alpha, \beta = 1, 2, 3, 4$.
2. Find the eigenstate and eigenvalue of the matrix h with the lowest eigenvalue. If one denotes this (normalized) eigenstate as $|\Psi\rangle$, you should be now able to express it as $|\Psi\rangle = \sum_{\alpha=1}^{4} A_\alpha|\phi_\alpha\rangle$ with known values of the coefficients A_α.
3. We are interested in the entanglement entropy of the state $|\Psi\rangle$ for a bipartition that divides sites 1 and 2. First consider a density matrix of the full system simply given by $|\Psi\rangle$,

$$\hat{\rho} = |\Psi\rangle\langle\Psi| \tag{3}$$

   To calculate the entanglement entropy between the 2-sites we first need to find the reduced density matrix, which we denote by $\hat{\rho}_1$, for this bipartition. Therefore, calculate

$$\hat{\rho}_1 = Tr_2(\hat{\rho}) \tag{4}$$

   where Tr_2 denotes a trace over the Hilbert space of site 2, that is over the 2-states $|\uparrow\rangle_2, |\downarrow\rangle_2$. This gives the reduced density matrix $\hat{\rho}_1$ for the subsystem consisting of site 1.
4. Re-express the density-matrix operator, $\hat{\rho}_1$ as a 2×2 matrix in the basis $|\uparrow\rangle_1, |\downarrow\rangle_1$, with matrix elements, for example, $\langle\uparrow_1|\hat{\rho}_1|\uparrow\rangle_1$ and so on.

5. Diagonalize your 2× 2 matrix representation of $\hat{\rho}_1$ to obtain its eigenvalues λ_i.
6. The von Neumann (bi-partite entanglement) entropy is defined as,

$$S_1^{vN} = -Tr_1[\hat{\rho}_1 \ln \hat{\rho}_1] = -\sum_i \lambda_i \ln \lambda_i \tag{5}$$

   Calculate S_1^{vN} as a function of h/J and make a sketch of S_1^{vN} versus h/J.
7. What is the value of S_1^{vN} as $\frac{h}{J} \to \infty$? As $\frac{h}{J} \to 0$? Explain the physics behind these two limits.

Solution 1

1. Using the fact that

$$\hat{\sigma}^x|\uparrow\rangle = |\downarrow\rangle,\;\; \hat{\sigma}^x|\downarrow\rangle = |\uparrow\rangle,\;\; \hat{\sigma}^z|\uparrow\rangle = |\uparrow\rangle,\;\; \hat{\sigma}^z|\downarrow\rangle = -|\downarrow\rangle \tag{6}$$

$$\hat{H}|\phi_1\rangle = -J|\phi_1\rangle - h(|\phi_3\rangle + |\phi_4\rangle), \tag{7}$$
$$\hat{H}|\phi_2\rangle = -J|\phi_2\rangle - h(|\phi_3\rangle + |\phi_4\rangle), \tag{8}$$
$$\hat{H}|\phi_3\rangle = J|\phi_3\rangle - h(|\phi_2\rangle + |\phi_1\rangle), \tag{9}$$
$$\hat{H}|\phi_4\rangle = J|\phi_4\rangle - h(|\phi_2\rangle + |\phi_1\rangle) \tag{10}$$

   Therefore in a matrix representation

$$\hat{H} = \begin{pmatrix} -J & 0 & -h & -h \\ 0 & -J & -h & -h \\ -h & -h & J & 0 \\ -h & -h & 0 & J \end{pmatrix} \tag{11}$$

2. The eivenvalues and corresponding eigenstates are as follows:

$$E_1 = -\Delta,\; |E_1\rangle = \frac{1}{\sqrt{2(\Delta - J)^2 + 8h^2}} \begin{pmatrix} 2h & 2h & \Delta - J & \Delta - J \end{pmatrix}^T, \tag{12}$$
$$E_2 = -J,\; |E_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}^T, \tag{13}$$
$$E_3 = J,\; |E_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 & 1 \end{pmatrix}^T, \tag{14}$$
$$E_4 = \Delta,\; |E_4\rangle = \frac{1}{\sqrt{2(\Delta + J)^2 + 8h^2}} \begin{pmatrix} -2h & -2h & J + \Delta & J + \Delta \end{pmatrix}^T \tag{15}$$

   where $\Delta = \sqrt{J^2 + 4h^2}$
   The ground state is $|E_1\rangle$ and

$$|\Psi\rangle = \frac{1}{\sqrt{2(\Delta - J)^2 + 8h^2}} \Big(2h(|\phi_1\rangle + |\phi_2\rangle) + (\Delta - J)(|\phi_3\rangle + |\phi_4\rangle)\Big). \tag{16}$$

3. From Eq. (16),

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\frac{1}{(\Delta - J)^2 + 4h^2}\Big(4h^2(|\phi_1\rangle + |\phi_2\rangle)(\langle\phi_1| + \langle\phi_1|) + 2h(\Delta - J)(|\phi_1\rangle + |\phi_2\rangle)(\langle\phi_3| + \langle\phi_4|)$$
$$+ 2h(\Delta - J)(|\phi_3\rangle + |\phi_4\rangle)(\langle\phi_1| + \langle\phi_2|) + (\Delta - J)^2(|\phi_3\rangle + |\phi_4\rangle)(\langle\phi_3| + \langle\phi_4|)\Big) \tag{17}$$

Using

$$Tr_2(|\phi_1\rangle\langle\phi_2) = Tr_2(|\phi_1\rangle\langle\phi_3|) = Tr_2(|\phi_3\rangle\langle\phi_4|) = Tr_2(|\phi_2\rangle\langle\phi_4|) = 0,$$
$$Tr_2(|\phi_1\rangle\langle\phi_1|) = |\uparrow\rangle_1\langle\uparrow|_1,\ Tr_2(|\phi_1\rangle\langle\phi_4|) = |\uparrow\rangle_1\langle\downarrow|_1,\ Tr_2(|\phi_3\rangle\langle\phi_3|) = |\uparrow\rangle_1\langle\uparrow|_1,$$
$$Tr_2(|\phi_2\rangle\langle\phi_2|) = |\downarrow\rangle_1\langle\downarrow|_1,\ Tr_2(|\phi_2\rangle\langle\phi_3|) = |\downarrow\rangle_1\langle\uparrow|_1,\ Tr_2(|\phi_4\rangle\langle\phi_4|) = |\downarrow\rangle_1\langle\downarrow|_1$$

and the fact that $(Tr_2(|\phi_i\rangle\langle\phi_j|))^\dagger = Tr_2(|\phi_j\rangle\langle\phi_i|)$, we can obtain

$$Tr_2(\rho) = \frac{1}{2}\frac{1}{(\Delta-J)^2+4h^2}\left[\Big(4h^2+(\Delta-J)^2\Big)\Big(|\uparrow\rangle_1\langle\uparrow|_1 + |\downarrow\rangle_1\langle\downarrow|_1\Big) + 4h(\Delta-J)\Big(|\uparrow\rangle_1\langle\downarrow|_1 + |\downarrow\rangle_1\langle\uparrow|_1\Big)\right] \tag{18}$$

4. In a matrix representation,

$$Tr_2(\rho) = \frac{1}{2}\frac{1}{(\Delta-J)^2+4h^2}\begin{pmatrix} 4h^2+(\Delta-J)^2 & 4h(\Delta-J) \\ 4h(\Delta-J) & 4h^2+(\Delta-J)^2 \end{pmatrix} \tag{19}$$

5. The eigenvalues and eigenstates of $Tr_2(\rho)$ are

$$\lambda_1 = \frac{1}{2}\frac{(2h-J+\Delta)^2}{4h^2+(J-\Delta)^2},\ |\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \tag{20}$$
$$\lambda_2 = \frac{1}{2}\frac{(2h+J-\Delta)^2}{4h^2+(J-\Delta)^2},\ |\lambda_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}. \tag{21}$$

6. Using the eigenvalues, the von-Neumann entropy is given by

$$\begin{aligned} S_1^{vN} = -\sum_i \lambda_i \ln\lambda_i &= -\frac{1}{2}\frac{(2h-J+\Delta)^2}{4h^2+(J-\Delta)^2}\ln\left(\frac{1}{2}\frac{(2h-J+\Delta)^2}{4h^2+(J-\Delta)^2}\right) - \frac{1}{2}\frac{(2h+J-\Delta)^2}{4h^2+(J-\Delta)^2}\ln\left(\frac{1}{2}\frac{(2h+J-\Delta)^2}{4h^2+(J-\Delta)^2}\right) \\ &= -\frac{(2x-1+\tilde{\Delta})^2}{4x^2+(1-\tilde{\Delta})^2}\ln|2x-1+\tilde{\Delta}| - \frac{(2x+1-\tilde{\Delta})^2}{4x^2+(1-\tilde{\Delta})^2}\ln|2x+1-\tilde{\Delta}| \\ &+ \ln 2 + \ln(4x^2+(1-\tilde{\Delta})^2) \end{aligned} \tag{22}$$

where $x = h/J$ and $\tilde{\Delta} = \sqrt{1+4x^2}$.

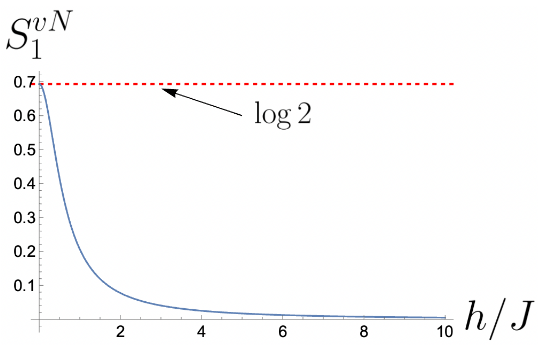

Figure 1: S_1^{vN} as a function of h/J.

7. (i) $\frac{h}{J} \to 0$ limit

$$\tilde{\Delta}(x) \to 1 \Rightarrow S_1^{vN} \to -2\ln 2x + \ln(2x^2) + \ln 2 = \ln 2 \tag{23}$$

In this limit, the Hamiltonian can be approximated into

$$\hat{H} \approx -J\hat{\sigma}_1^z\hat{\sigma}_2^z \tag{24}$$

and the corresponding ground state is singlet state which is maximally entangled state. Therefore the entanglement entropy is given by $\ln 2$.

(ii) $\frac{h}{J} \to \infty$ limit

$$\tilde{\Delta}(x) \to 2x \Rightarrow S_1^{vN} \to 2\ln 4x + \ln 8x^2 + \ln 2 = 0 \tag{25}$$

In this limit, the Hamiltonian can be approximated into

$$\hat{H} \approx -h(\hat{\sigma}_1^x + \hat{\sigma}_2^x) \tag{26}$$

and the corresponding ground state is product state($|\uparrow\rangle_1 \otimes |\downarrow\rangle_2$). As a result, the entanglement entropy is zero.

⋆ Problem 2 ⋆ Operator in Heisenberg picture

Consider a harmonic oscillator with the Hamiltonian

$$H = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) \tag{27}$$

Where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators respectively. In the Heisenberg picture, in contrast to the Schrödinger picture, the states are time-independent, but the operators for them are time-dependent.

1. Use the Heisenberg equation of motion to calculate the time evolution of the operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.
2. Calculate the time evolution of the operators $\hat{x}(t)$ and $\hat{p}(t)$.
3. Use the results of the previous parts of the task to calculate the commutators $[\hat{x}(t_1), \hat{x}(t_2)]$ and $[\hat{x}(t_1), \hat{p}(t_2)]$ of the now time-dependent operators.
4. At time $t = 0$, the system is in the ground state $|\psi(0)\rangle = |0\rangle$. Calculate the correlation function $\langle\hat{x}(0)\hat{x}(t)\rangle$.

Solution 2

The Heisenberg equation of motion for an operator $\hat{A}(t)$ is

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}(t)] + \frac{\partial\hat{A}(t)}{\partial t} \tag{28}$$

where the $\hat{H}$ is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\Big(a^\dagger a + \frac{1}{2}\Big). \tag{29}$$

1. The equations of motion for $a(t)$ and $a^\dagger(t)$ are given by

$$\frac{da}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}] = i\omega[\hat{a}, \hat{a}^\dagger\hat{a}] = -i\omega\hat{a} \Rightarrow \hat{a}(t) = e^{-i\omega t}a(0), \tag{30}$$

$$\frac{da^\dagger}{dt} = \frac{i}{\hbar}[\hat{a^\dagger}, \hat{H}] = i\omega[\hat{a}^\dagger, \hat{a}^\dagger\hat{a}] = i\omega\hat{a} \Rightarrow \hat{a}^\dagger(t) = e^{i\omega t}a^\dagger(0), \tag{31}$$

(32)

2. $\hat{x}(t)$ and $\hat{p}(t)$ are obtained from the above result:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{m\omega}}(\hat{a}^\dagger(t) + \hat{a}(t)) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t, \tag{33}$$

$$\hat{p}(t) = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger(t) - \hat{a}(t)) = p(0)\cos\omega t - m\omega x(0)\sin\omega t. \tag{34}$$

The operators satisfy the classical equations of motion.

The same results can be obtained by solving the system of equations

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{x}(t)] = \frac{\hat{p}(t)}{m}, \tag{35}$$

$$\frac{d\hat{p}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{p}(t)] = -m\omega^2\hat{x}(t) \tag{36}$$

3. Using the above results, we can obtain following values of commutators $[\hat{x}(t_1), \hat{x}(t_2)]$ and $[\hat{x}(t_1), \hat{p}(t_2)]$:

$$\begin{aligned}
[\hat{x}(t_1), \hat{x}(t_2)] &= \frac{1}{m\omega}\cos\omega t_1 \sin\omega t_2[\hat{x}(0), \hat{p}(0)] + \frac{1}{m\omega}\sin\omega t_1\cos\omega t_2[\hat{p}(0), \hat{x}(0)] \\
&= \frac{i\hbar}{m\omega}\sin\omega(t_2 - t_1),
\end{aligned}\tag{37}$$

$$\begin{aligned}
[\hat{x}(t_1), \hat{p}(t_2)] &= \cos\omega t_1\cos\omega t_2[\hat{x}(0), \hat{p}(0)] - \sin\omega t_1\sin\omega t_2[\hat{p}(0), \hat{x}(0)] \\
&= i\hbar\cos\omega(t_2 - t_1)
\end{aligned}\tag{38}$$

4. The system is in the ground state $|\psi(0)\rangle = |0\rangle$ at $t = 0$. To calculate the correlation function we switch back to the Heisenberg picture where $\hat{x}(0) = \hat{x}$.

$$\begin{aligned}
\langle\hat{x}(0)\hat{x}(t)\rangle &= \langle 0|\hat{x}e^{i\hat{H}t/\hbar}\hat{x}e^{-i\hat{H}t/\hbar}|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(\hat{a}^\dagger + \hat{a})e^{i\hat{H}t/\hbar}(\hat{a}^\dagger + \hat{a})|0\rangle e^{-i\omega t/2} \\
&= \frac{\hbar}{2m\omega}\langle 1|e^{i\hbar Ht/\hbar}|1\rangle e^{-i\omega t/2} = \frac{\hbar}{2m\omega}e^{i\omega t}
\end{aligned}\tag{39}$$

Problem 3 Spin in a time-dependent magnetic field

In the lecture, the Larmor precession of the electron spin in a static magnetic field in the z-direction $\mathbf{B} = B\hat{z}$ was discussed. In the following, the behavior of the spin-$\frac{1}{2}$ particle will be investigated when a periodic magnetic field is additionally applied in the xy-plane.

Consider an electron spin in a time-dependent external magnetic field:

$$\mathbf{B}(t) = B_1\cos\omega t\hat{e}_x + B_1\sin\omega t\hat{e}_y + B_0\hat{e}_z \tag{40}$$

1. Write the Hamiltonian of the system explicitly in matrix form.

2. Show that the problem can be transformed into the problem of an electron in a static magnetic field $\bar{\mathbf{B}} = B_x\hat{e}_x + B_z\hat{e}_y$ by using a sutiable transformation:

$$\psi_\uparrow(t) = a(t)\bar{\psi}_\uparrow(t), \quad \psi_\downarrow(t) = b(t)\bar{\psi}_\downarrow(t). \tag{41}$$

3. Use the time evolution operator $\hat{U}(t, 0)$ to calculate the time evolution of the state $|\bar{\psi}(t)\rangle$ of the electron spin in the effective static magnetic field $\bar{\mathbf{B}}$.

4. At time $t = 0$ the particle is in the state $|\psi(0)\rangle = |\uparrow\rangle$. Calculate the probability $P_\downarrow(t)$ of finding the spin at time t in the state $|\downarrow\rangle$. For which frequency ω of the external field and after which time does the probability become maximum?

Solution 3

1. The Hamiltonian in the matrix representation is given by

$$\hat{H} = \mu_B \sigma \cdot \mathbf{B} = \mu_B B_1 \left[\cos\omega t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\omega t \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] + \mu_B B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{42}$$

If we define the frequencies $\omega = 2\mu_B B$ it can be written compactly as

$$\hat{H} = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \tag{43}$$

2. From the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \tag{44}$$

we obtain the equations of motion for the components $\psi_\uparrow(t)$ and $\psi_\downarrow(t)$

$$i\frac{\partial \psi_\uparrow(t)}{\partial t} = \frac{\omega_0}{2}\psi_\uparrow + \frac{\omega_1}{2} e^{-i\omega t}\psi_\downarrow, \tag{45}$$

$$i\frac{\partial \psi_\downarrow}{\partial t} = \frac{\omega_1}{2} e^{i\omega t}\psi_\uparrow - \frac{\omega_0}{2}\psi_\downarrow \tag{46}$$

Now let us consider the following transformation:

$$\psi_\uparrow(t) = e^{-i\omega t/2}\bar{\psi}_\uparrow(t), \quad \psi_\downarrow(t) = e^{i\omega t/2}\bar{\psi}_\downarrow(t) \tag{47}$$

Then the equations of the motions in terms of the $\bar{\psi}_{\uparrow,\downarrow}$ are given by

$$i\frac{\partial \bar{\psi}_\uparrow}{\partial t} = -\frac{\omega - \omega_0}{2}\psi_\uparrow + \frac{\omega_1}{2}\psi_\downarrow, \tag{48}$$

$$i\frac{\partial \bar{\psi}_\downarrow}{\partial t} = \frac{\omega_1}{2}\psi_\uparrow + \frac{\omega - \omega_0}{2}\psi_\downarrow. \tag{49}$$

We have now transformed the problem to a system with the effective Hamiltonian

$$\hat{\bar{H}} = \frac{\hbar}{2} \begin{pmatrix} -(\omega - \omega_0) & \omega_1 \\ \omega_1 & \omega - \omega_0 \end{pmatrix} \tag{50}$$

This corresponds to an electron spin in the static magnetic field $\bar{\mathbf{B}} = (B_1, 0, \frac{\Delta\omega}{2\mu_B})$ with $\Delta\omega = \omega - \omega_0$. The transformation (47) can also be written in matrix form as follows:

$$|\psi\rangle = e^{-i\omega t\sigma_z/2}|\bar{\psi}\rangle \tag{51}$$

It is nothing but a time-dependent rotation in spin space around the z-axis. We have switched to a rotating reference system and follow the rotation of the magnetic field $B_1(t)$.

3. To determine the time evolution of the state $|\bar{\psi}(t)\rangle$, we apply the time evolution operator of $\hat{\bar{H}}$.

$$\bar{U}(t,t') = e^{-\frac{i}{\hbar}(t-t')\hat{\bar{H}}}, \tag{52}$$

$$|\bar{\psi}(t)\rangle = e^{-\frac{i}{\hbar}t\hat{\bar{H}}}|\bar{\psi}(0)\rangle = e^{-i(\omega_1\sigma_x/2 - \Delta\omega\sigma_z/2)t}|\bar{\psi}(0)\rangle$$

Using the fact that

$$e^{i\sigma\cdot\hat{n}\theta} = \cos\theta + i\sigma\cdot\hat{n}\sin\theta \tag{53}$$

$$|\bar{\psi}\rangle = \left(\cos\frac{\Omega}{2} - i\frac{1}{\Omega}(\omega_1\sigma_x - \Delta\omega\sigma_z)\sin\frac{\Omega t}{2}\right)|\bar{\psi}(0)\rangle \tag{54}$$

where $\Omega = \sqrt{\omega_1^2 + \Delta\omega^2}$.

4. The amplitude for a spin flip from $|\psi(0)\rangle = |\uparrow\rangle$ to $|\psi(t)\rangle = |\downarrow\rangle$ is

$$\begin{aligned}\langle\downarrow|\psi(t)\rangle &= \langle\downarrow e^{-i\omega t\sigma_z/2}|\bar{\psi}(t)\rangle = e^{i\omega t/2}\langle\downarrow|\left(\cos\frac{\Omega t}{2} - i\frac{1}{\Omega}(\omega_1\sigma_x - \Delta\omega\sigma_z)\sin\frac{\Omega t}{2}\right)|\uparrow\rangle \\ &= -ie^{i\omega t/2}\frac{\omega_1}{\Omega}\sin\frac{\Omega t}{2}, \end{aligned} \tag{55}$$

where we used $|\bar{\psi}(0)\rangle = |\psi(0)\rangle = |\uparrow\rangle$. We obtain the probability

$$P_\downarrow(t) = |\langle\downarrow|\psi(t)\rangle|^2 = \frac{\omega_1^2}{2\Omega^2}(1 - \cos\Omega t). \tag{56}$$

The probability $P_\downarrow(t)$ reaches the maximum value

$$\frac{\omega_1^2}{\Omega^2} = \frac{\omega_1^2}{(\omega - \omega_0)^2 + \omega_1^2} \tag{57}$$

after a π-pulse, $t = \frac{\pi}{\Omega}$.

For a resonant magnetic field $\omega = \omega_0$, this has the largest value with $P_\downarrow(\pi/\Omega) = 1$, and the spin then changes with certainty over time.