Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

Summer Semester 2024
Exercise Sheet 13

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Magnetic perturbation

Consider a spin-1/2 particle in a large magnetic field oriented along $\hat{\boldsymbol{z}}$:

$$H_0 = -\boldsymbol{\mu} \cdot B_0 \hat{\boldsymbol{z}}, \tag{1}$$

where $\boldsymbol{\mu} = -\gamma \boldsymbol{S}$ is the magnetic dipole moment, γ is the gyromagnetic ratio, and $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is a vector of spin operators. Let us consider the effects of the perturbation

$$V = -\boldsymbol{\mu} \cdot (B_1 \hat{\boldsymbol{z}} + B_2 \hat{\boldsymbol{x}}). \tag{2}$$

1. Using perturbation theory formulas derived during the lectures, find the changes in the eigen-energies to lowest order in B_1 and B_2.
2. Likewise, find the change in the eigenstates to lowest order in B_1 and B_2.
3. Calculate the exact eigen-energies and eigenvectors of $H = H_0 + V$ and compare them with the results of parts 1 and 2.

⋆ Problem 2 ⋆ Schwinger representation of angular momentum operators

Consider two decoupled harmonic oscillators with lowering operators a_+ and a_-. They obey

$$[a_+, a_+^\dagger] = 1, \tag{3}$$
$$[a_-, a_-^\dagger] = 1 \tag{4}$$

and commute with each other, $[a_+, a_-] = [a_+, a_-^\dagger] = [a_+^\dagger, a_-] = [a_+^\dagger, a_-^\dagger] = 0$.

1. Let us suppose that in the Hamiltonian

$$H = \hbar\omega_+(a_+^\dagger a_+ + \tfrac{1}{2}) + \hbar\omega_-(a_-^\dagger a_- + \tfrac{1}{2}) \tag{5}$$

the two harmonic oscillators have the same frequency $\omega_+ = \omega_- = \omega_0$. What is the degeneracy of an arbitrary state $|n_+, n_-\rangle$? List all the states with the same energy.

2. Now consider the operators

$$J_+ = \hbar a_+^\dagger a_-, \tag{6}$$

$$J_- = \hbar a_-^\dagger a_+. \tag{7}$$

Show that $[H, J_\pm] = 0$. Give a physical explanation for why $J_\pm$ preserves the energy.

3. Define the operator

$$J_z \equiv \frac{1}{2\hbar}[J_+, J_-]. \tag{8}$$

Find it and show that

$$[J_z, J_\pm] = \pm\hbar J_\pm. \tag{9}$$

4. Introduce

$$J_x \equiv \frac{1}{2}(J_+ + J_-), \tag{10}$$

$$J_y \equiv \frac{1}{2\mathrm{i}}(J_+ - J_-) \tag{11}$$

and express

$$\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2 \tag{12}$$

in terms of $N = a_+^\dagger a_+ + a_-^\dagger a_-$. Compare with results of part 1.

There will be no Problem 3. Happy summer holidays!