
Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 13

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Magnetic perturbation

Consider a spin-1/2 particle in a large magnetic field oriented along $\hat{\boldsymbol{z}}$:

$$H_0 = -\boldsymbol{\mu} \cdot B_0 \hat{\boldsymbol{z}}, \tag{1}$$

where $\boldsymbol{\mu} = -\gamma \boldsymbol{S}$ is the magnetic dipole moment, γ is the gyromagnetic ratio, and $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is a vector of spin operators. Let us consider the effects of the perturbation

$$V = -\boldsymbol{\mu} \cdot (B_1 \hat{\boldsymbol{z}} + B_2 \hat{\boldsymbol{x}}). \tag{2}$$

1. Using perturbation theory formulas derived during the lectures, find the changes in the eigen-energies to lowest order in B_1 and B_2.
2. Likewise, find the change in the eigenstates to lowest order in B_1 and B_2.
3. Calculate the exact eigen-energies and eigenvectors of $H = H_0 + V$ and compare them with the results of parts 1 and 2.

Solution 1

1. The bare Hamiltonian we write as

$$H_0 = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix},$$

where $\omega_0 = \gamma B_0$ is the Larmor frequency. Clearly, it has the energies and eigenvectors:

$$E_1^{(0)} = +\tfrac{1}{2}\hbar\omega_0, \qquad |1\rangle^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$E_2^{(0)} = -\tfrac{1}{2}\hbar\omega_0, \qquad |2\rangle^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The perturbation we write as:

$$V = \frac{\hbar}{2}\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 \end{pmatrix},$$

where $\omega_1 = \gamma B_1$ and $\omega_2 = \gamma B_2$. The energy perturbation formula

$$E_n = E_n^{(0)} + {}^0\langle n|V|n\rangle^0 + \sum_{m\neq n} \frac{\left|{}^0\langle m|V|n\rangle^0\right|^2}{E_n^0 - E_m^0} + \cdots$$

gives

$$\begin{aligned}
E_1 &= +\tfrac{1}{2}\hbar\omega_0 + \tfrac{1}{2}\hbar\omega_1 + \frac{(\frac{1}{2}\hbar\omega_2)^2}{\hbar\omega_0} + \cdots \\
E_2 &= -\tfrac{1}{2}\hbar\omega_0 - \tfrac{1}{2}\hbar\omega_1 - \frac{(\frac{1}{2}\hbar\omega_2)^2}{\hbar\omega_0} + \cdots
\end{aligned}$$

2. The eigenvector perturbation formula

$$|n\rangle = |n\rangle^{(0)} + \sum_{m\neq n} \frac{{}^0\langle m|V|n\rangle^0}{E_n^0 - E_m^0} |m\rangle^{(0)} + \cdots$$

gives

$$\begin{aligned}
|1\rangle^{(0)} &= \begin{pmatrix} 1 \\ \frac{\omega_2}{2\omega_0} \end{pmatrix} + \cdots \\
|2\rangle^{(0)} &= \begin{pmatrix} -\frac{\omega_2}{2\omega_0} \\ 1 \end{pmatrix} + \cdots
\end{aligned}$$

3. The exact solution is

$$\begin{aligned}
E_{1/2} &= \pm\tfrac{1}{2}\hbar\Omega, \\
|1\rangle &= \frac{1}{\sqrt{2\Omega(\Omega+\omega_0+\omega_1)}} \begin{pmatrix} \Omega+\omega_0+\omega_1 \\ \omega_2 \end{pmatrix}, \\
|2\rangle &= \frac{1}{\sqrt{2\Omega(\Omega+\omega_0+\omega_1)}} \begin{pmatrix} -\omega_2 \\ \Omega+\omega_0+\omega_1 \end{pmatrix},
\end{aligned}$$

where

$$\Omega = \sqrt{(\omega_0+\omega_1)^2 + \omega_2^2} = \omega_0 + \omega_1 + \frac{\omega_2^2}{2\omega_0} + \cdots$$

This agrees with the previous after a Taylor expansion.

⋆ Problem 2 ⋆ Schwinger representation of angular momentum operators

Consider two decoupled harmonic oscillators with lowering operators a_+ and a_-. They obey

$$[a_+, a_+^\dagger] = 1, \tag{3}$$
$$[a_-, a_-^\dagger] = 1 \tag{4}$$

and commute with each other, $[a_+, a_-] = [a_+, a_-^\dagger] = [a_+^\dagger, a_-] = [a_+^\dagger, a_-^\dagger] = 0$.

1. Let us suppose that in the Hamiltonian

$$H = \hbar\omega_+(a_+^\dagger a_+ + \tfrac{1}{2}) + \hbar\omega_-(a_-^\dagger a_- + \tfrac{1}{2}) \tag{5}$$

the two harmonic oscillators have the same frequency $\omega_+ = \omega_- = \omega_0$. What is the degeneracy of an arbitrary state $|n_+, n_-\rangle$? List all the states with the same energy.

2. Now consider the operators

$$J_+ = \hbar a_+^\dagger a_-, \tag{6}$$
$$J_- = \hbar a_-^\dagger a_+. \tag{7}$$

Show that $[H, J_\pm] = 0$. Give a physical explanation for why $J_\pm$ preserves the energy.

3. Define the operator

$$J_z \equiv \frac{1}{2\hbar}[J_+, J_-]. \tag{8}$$

Find it and show that

$$[J_z, J_\pm] = \pm\hbar J_\pm. \tag{9}$$

4. Introduce

$$J_x \equiv \frac{1}{2}(J_+ + J_-), \tag{10}$$
$$J_y \equiv \frac{1}{2\mathrm{i}}(J_+ - J_-) \tag{11}$$

and express

$$\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2 \tag{12}$$

in terms of $N = a_+^\dagger a_+ + a_-^\dagger a_-$. Compare with results of part 1.

Solution 2

1. The energy of the state $|n_+, n_-\rangle$ is then

$$E_{n_+,n_-} = \hbar\omega_0(n_+ + n_- + 1),$$

which is the same as long as $N = n_+ + n_-$ is the same. $n_\pm$ must be positive or zero so the degenerate states are

$$|N, 0\rangle, |N-1, 1\rangle, |N-2, 2\rangle, \ldots, |1, N-1\rangle, |0, N\rangle$$

and the overall degree of degeneracy is $N + 1$.

2. Write

$$H = \hbar\omega_0(N_+ + N_- + 1).$$

Then

$$[H, a_\pm] = \hbar\omega_0[N_+ + N_- + 1, a_\pm] = -\hbar\omega_0 a_\pm,$$
$$[H, a_\pm^\dagger] = \hbar\omega_0[N_+ + N_- + 1, a_\pm^\dagger] = +\hbar\omega_0 a_\pm^\dagger.$$

Hence

$$[H, J_+] = [H, \hbar a_+^\dagger a_-] = \hbar[H, a_+^\dagger]a_- + \hbar a_+^\dagger[H, a_-] = \hbar^2\omega_+ a_+^\dagger a_- - \hbar^2\omega_- a_+^\dagger a_- = 0$$

because $\omega_+ = \omega_-$, and likewise for $[H, J_-] = 0$. Physically, $J_\pm$ move a quantum from one harmonic oscillator to the other. As long as the frequencies of the two harmonic oscillators are the same, this preserves the total energy.

3. After a little algebra, one readily finds that

$$J_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{\hbar}{2}(N_+ - N_-).$$

Hence

$$\begin{aligned}
[J_z, J_+] &= \frac{\hbar^2}{2}[N_+ - N_-, a_+^\dagger a_-] \\
&= \frac{\hbar^2}{2}([N_+ - N_-, a_+^\dagger]a_- + a_+^\dagger[N_+ - N_-, a_-]) \\
&= \hbar^2 a_+^\dagger a_- = \hbar J_+, \\
[J_z, J_-] &= \frac{\hbar^2}{2}[N_+ - N_-, a_-^\dagger a_+] \\
&= \frac{\hbar^2}{2}([N_+ - N_-, a_-^\dagger]a_+ + a_-^\dagger[N_+ - N_-, a_+]) \\
&= -\hbar^2 a_-^\dagger a_+ = -\hbar J_-.
\end{aligned}$$

4. So

$$\begin{aligned}
J_x &= \frac{\hbar}{2}(a_+^\dagger a_- + a_-^\dagger a_+), \\
J_y &= \frac{\hbar}{2\mathrm{i}}(a_+^\dagger a_- - a_-^\dagger a_+)
\end{aligned}$$

and therefore

$$\begin{aligned}
\boldsymbol{J}^2 &= \frac{\hbar^2}{4}\Big\{(a_+^\dagger a_- + a_-^\dagger a_+)^2 - (a_+^\dagger a_- - a_-^\dagger a_+)^2 + (a_+^\dagger a_+ - a_-^\dagger a_-)^2\Big\} \\
&= \frac{\hbar^2}{4}\Big\{a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_- - (-a_+^\dagger a_- a_-^\dagger a_+ - a_-^\dagger a_+ a_+^\dagger a_-) + (N_+ - N_-)^2\Big\} \\
&= \frac{\hbar^2}{4}\big\{2N_+(1 + N_-) + 2(1 + N_+)N_- + N_+^2 + N_-^2 - 2N_+N_-\big\} \\
&= \frac{\hbar^2}{4}\big\{N_+^2 + N_-^2 + 2N_+N_- + 2(N_+ + N_-)\big\} = \hbar^2\frac{N}{2}\left(\frac{N}{2} + 1\right).
\end{aligned}$$

So the effective $j = N/2$. The corresponding degeneracy $2j + 1 = N + 1$ agrees with part 1.

There will be no Problem 3. Happy summer holidays!