Moderne Theoretische Physik I
Grundlagen der Quantenmechanik

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Exercise Sheet 7

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The problems whose solutions you need to upload are designated with stars.

⋆ Problem 1 ⋆ Reflection and transmission for a Dirac delta potential

Consider a particle of mass m subject to the potential

$$V(x) = V_0\,\delta(x) \tag{1}$$

where V_0 can be positive or negative.

1. Assume that the particle is coming from the left ($x < 0$) with a momentum $\hbar k$, i.e., assume that $\psi(x)$ has a $\mathrm{e}^{\mathrm{i}kx}$ component as $x \to -\infty$, but does not have a $\mathrm{e}^{-\mathrm{i}kx}$ component as $x \to +\infty$. Find the energy and the corresponding wavefunction $\psi(x)$.
2. Define the reflection coefficient as
$$R \equiv \lim_{x\to-\infty} \left| \frac{j_{\text{reflected}}}{j_{\text{incoming}}} \right| \tag{2}$$
and the transmission coefficient as
$$T \equiv \lim_{x\to\infty} \left| \frac{j_{\text{transmitted}}}{j_{\text{incoming}}} \right|, \tag{3}$$
where j is the current ($\partial_t |\psi|^2 + \partial_x j = 0$). Calculate them for the $\psi(x)$ of part 1. Check that $R + T = 1$.

Next, let us put a wall a distance L in front of the Dirac delta well:

$$V(x) = \begin{cases} V_0\,\delta(x), & \text{when } x \leq L, \\ \infty, & \text{when } x > L. \end{cases} \tag{4}$$

3. For this new potential, find the wavefunction $\psi(x)$ for a particle incoming from the left with momentum $\hbar k$.
4. Explicitly calculate the reflection coefficient for the $\psi(x)$ of part 3. Discuss.

⋆ Problem 2 ⋆ Orbital angular momentum

In Hamiltonian mechanics, the Poisson bracket between two phase-space functions $f(q_i, p_i)$ and $g(q_i, p_i)$ is defined as

$$\{f, g\} \equiv \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{5}$$

1. Calculate the Poisson bracket $\{L_i, L_j\}$ between the orbital angular momentum operators $L_i = \sum_{jk} \epsilon_{ijk} x_j p_k$, that is $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$. (Hint: use $\{f, gh\} = \{f, g\}h + g\{f, h\}$.)

2. Next, evaluate $\{L_i, \boldsymbol{L}^2\}$, $\{L_\pm, L_z\}$, and $\{L_+, L_-\}$, where $\boldsymbol{L}^2 \equiv L_x^2 + L_y^2 + L_z^2$ and $L_\pm \equiv L_x \pm \mathrm{i}L_y$.

3. If L_x and L_y are conserved quantities relative to some Hamiltonian H, show that L_z must also be conserved. (Hint: exploit a fundamental property of the Poisson bracket.)

According to the canonical quantization procedure, if $f(x_i, p_i)$ is a function on phase space, then in quantum mechanics it becomes the operator $\hat{f} \equiv f(\hat{x}_i, \hat{p}_i)$, with the Poisson brackets moreover mapping onto commutators according to

$$[\hat{f}, \hat{g}] = \mathrm{i}\hbar\,\widehat{\{f, g\}}. \tag{6}$$

Thus $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\,\widehat{\{x_i, p_j\}} = \mathrm{i}\hbar\delta_{ij}\hat{1} = \mathrm{i}\hbar\delta_{ij}$, for instance.

4. Use this to find $[\hat{L}_i, \hat{L}_j]$, $[\hat{L}_i, \hat{\boldsymbol{L}}^2]$, $[\hat{L}_\pm, \hat{L}_z]$, and $[\hat{L}_+, \hat{L}_-]$ from the previous parts of this problem.

5. If a state $|\phi\rangle$ has

$$\hat{\boldsymbol{L}}^2 |\phi\rangle = \alpha |\phi\rangle, \qquad L_z |\phi\rangle = \mu |\phi\rangle, \tag{7}$$

   what are the corresponding eigenvalues of $|\phi_+\rangle \equiv \hat{L}_+ |\phi\rangle$ and $|\phi_-\rangle \equiv \hat{L}_- |\phi\rangle$?

Problem 3 Particle in a finite-depth well in 3 dimensions

Consider a particle described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + V_0\Theta(\hat{x}^2 + \hat{y}^2 + \hat{z}^2 - R^2) \tag{8}$$

where $\boldsymbol{r} = (x, y, z)$ are the Cartesian coordinates, $\hat{p}_x = -\mathrm{i}\hbar\partial_x$, $\hat{p}_y = -\mathrm{i}\hbar\partial_y$, and $\hat{p}_z = -\mathrm{i}\hbar\partial_z$ are the momentum operators, and $\Theta(x)$ is the Heaviside step function. $V_0 > 0$ is the potential well depth and $R > 0$ is its radial size.

1. Write the Hamiltonian as a differential operator in spherical coordinates (in the position representation). Express the angular part in terms of the differential operator $\hat{\boldsymbol{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, where $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$ is the angular momentum operator.

2. Next, assume that you are given a wavefunction $Y(\theta, \phi)$ that depends on the spherical angles θ and ϕ and that is an eigenvector of the $\hat{\boldsymbol{L}}^2$ operator with an eigenvalue $\lambda > 0$:

$$\hat{\boldsymbol{L}}^2 Y(\theta, \phi) = \hbar^2\lambda Y(\theta, \phi). \tag{9}$$

   If a stationary state of energy E has the form $\psi(\boldsymbol{r}) = \psi(r, \theta, \phi) = \varphi(r)Y(\theta, \phi)$, how does the corresponding radial stationary Schrödinger equation for $\varphi(r)$ look like?

3. Now consider the special case when $Y(\theta, \phi) = 1$. What is the value of λ? Solve the radial stationary Schrödinger equation for this case and find all the bound states that do not depend on the spherical angles. If any transcendental equations arise, formulate their solutions graphically.

4. Does a bounded state always exist? If not, how large does V_0 have to be?