KARLSRUHER INSTITUT FÜR TECHNOLOGIE

INSTITUT FÜR THEORIE
DER KONDENSIERTEN MATERIE

Moderne Theoretische Physik II (WS 2024/25)

Prof. Dr. A. Shnirman
Adrian Reich

Exam No. 1
18.02.2025

---

1. Time-dependent perturbation

(10 points)

The Hamiltonian operator of a one-dimensional harmonic oscillator is

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \tag{1}$$

Additionally, let there be a time-dependent perturbation $V(t)$ with

$$V(t) = \lambda\sqrt{2m\hbar\omega^3}\, x\, e^{-|t|/\tau} \tag{2}$$

where $\lambda \ll 1$ and $\tau > 0$ is a time scale. The total Hamiltonian operator is thus $H(t) = H_0 + V(t)$. For $t \to -\infty$, the system is in its ground state ($n = 0$).

a) In first-order perturbation theory in λ, determine the probability with which the system is in its first excited state ($n = 1$) at $t \to +\infty$. (6 points)

b) In **second**-order perturbation theory in λ, determine the probability with which the system is in its second excited state ($n = 2$) at $t \to +\infty$. (4 points)

Note: It is useful to express the Hamiltonian operator via raising and lowering operators, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{i}{m\omega}p)$, $a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{i}{m\omega}p)$.

2. Dirac equation

(10 points)

The Dirac equation is given by

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0 \tag{3}$$

with the four Dirac matrices γ^μ, which satisfy the following algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{4}$$

$g^{\mu\nu}$ is the Minkowski metric.

a) Show that every solution Ψ of the Dirac equation also solves the Klein-Gordon equation $(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\Psi = 0$. (3 points)

b) Show that $\partial_\mu j^\mu_A \propto i\bar{\Psi}\gamma_5\Psi$ with $j^\mu_A = c\bar{\Psi}\gamma^\mu\gamma_5\Psi$ and determine the proportionality constant. It is $\bar{\Psi} = \Psi^\dagger\gamma^0$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ satisifies $\{\gamma^5, \gamma^\mu\} = 0$. (3 points)

c) Determine (up to normalization constants) the four linearly independent solutions of the Dirac equation at a given 3-momentum $\vec{p}$. (4 Punkte)

3. Canonical partition function **(10 points)**

We consider a system with three possible single-particle states. The energies of the ~~the~~ single-particle states are spin-independent and given by $\varepsilon_1 = \varepsilon_2 = 0$ as well as $\varepsilon_3 = \Delta$. Now, we introduce two (indistinguishable) spin-$\frac{1}{2}$ fermions into the system.

a) Determine the canonical partition function. Remember to take into account the spin degeneracy. (6 points)

b) Determine the entropy $S(T)$ of the system as well as the expectation value of the energy $U(T)$ as a function of the temperature T. For each of these quantities, find the limits $T \to 0$ and $T \to \infty$. (4 points)

4. Bose-Einstein condensation **(10 points)**

We consider non-interacting bosons in a 3-dimensional volume V with the dispersion relation $\varepsilon(\vec{p}) = \alpha|\vec{p}|^\kappa$ $(\alpha, \kappa > 0)$ at a temperature $T > 0$.

a) Calculate the density of states $\nu(\varepsilon)$. (3 points)

b) Using the density of states, give an expression for the particle number density $n(T, V, \mu)$ and determine the values of κ for which at low temperatures Bose-Einstein condensation takes place. (7 points)
Note: The occuring integrals do not need to be solved explicitly.