

# Moderne Theoretische Physik II (Quantenmechanik II und Statistik)

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## Aufgabe 1: (\*) Gesamtdrehimpuls (2 Punkte)

Die Dirac Gleichung kann in folgender Form geschrieben werden

$$i\partial_t\psi = H\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi.$$

Zeigen Sie, dass der Gesamtdrehimpuls  $\vec{J} = \vec{L} + \vec{S}$  mit dem Hamiltonian kommutiert, also  $[H, \vec{J}] = 0$ , wobei der Spin-Operator  $\vec{S}$  gegeben ist durch

$$\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

Solution

We calculate first the commutators

$$[H, L^k] = \varepsilon^{knm}[\alpha^j p^j, x^n p^m] = -i\varepsilon^{kjm} \alpha^j p^m$$

and

$$[H, S^k] = [\alpha^j p^j + \beta m, S^k] = i\varepsilon^{jkl} \alpha^l p^j,$$

where we used

$$[\alpha^j, S^k] = i\varepsilon^{jkl} \alpha^l$$

and

$$[\beta, S^k] = 0.$$

In total we obtain

$$[H, J^k] = [H, L^k] + [H, S^k] = -i\varepsilon^{kjm} \alpha^j p^m + i\varepsilon^{jkl} \alpha^l p^j = 0.$$

## Aufgabe 2: (\*) Projektoren für Energie und Spin (1 + 1 + 1 + 1 = 4 Punkte)

a) Berechnen Sie die Kommutatoren

$$[\Lambda_{\pm}(p), \Sigma(s)],$$

wobei  $\Lambda_{\pm}(p) = \frac{\pm\not{p} + m}{2m}$  und  $\Sigma(s) = \frac{1 + \gamma_5 \not{s}}{2}$  den Energie- bzw. Spinprojektor bezeichnet.  $s^\mu$  ist der Spin-Vierervektor und  $p^\mu$  der Viererimpuls.

Solution

We calculate the commutator

$$[\Lambda_{\pm}(p), \Sigma(s)] = \mp \frac{\gamma_5 \not{p} \cdot s}{2m}$$

- b) Zerlegen Sie  $s^\mu$  in  $\xi p^\mu + \eta g^{\mu 0}$ , indem Sie  $s^2 = -1$  und  $s \cdot p = 0$  benutzen. Gegen welchen Ausdruck strebt  $s^\mu$  für  $|\vec{p}| \rightarrow \infty$ ?

Solution

We consider first the condition  $s^2 = -1$ :

$$s^2 = -1 = s_\mu s^\mu = \xi^2 m^2 + 2\xi \eta p_0 + \eta^2 g_0^0$$

We use the second condition to obtain an expression of  $\eta$  as a function of  $\xi$ :

$$\begin{aligned} s \cdot p = 0 &= \xi p^\mu p_\mu + \eta p^\mu g_{\mu 0} = \xi m^2 + \eta p_0 \\ \rightarrow \eta &= -\xi \frac{m^2}{p_0} \end{aligned}$$

Inserting this into the first equations leads to

$$\begin{aligned} -1 &= \xi^2 m^2 - 2\xi^2 m^2 + \xi^2 \frac{m^4}{p_0^2} \\ \rightarrow \xi &= \frac{p_0}{m \sqrt{p_0^2 - m^2}}, \end{aligned}$$

where we chose the positive solution for  $\xi$ . In the limit  $|\vec{p}| \rightarrow \infty$  we have  $p_0^2 \gg m^2$  and obtain

$$\begin{aligned} \xi &\rightarrow \frac{1}{m}, \\ \eta &\rightarrow 0. \end{aligned}$$

- c) Berechnen Sie  $\Lambda_{\pm} \Sigma(s)$  für  $|\vec{p}| \rightarrow \infty$ .

Solution

$$\left( \frac{\pm \not{p} + m}{2m} \right) \left( \frac{1 + \gamma_5 \not{s}}{2} \right) = \frac{1}{4} \pm \frac{\not{p} \gamma_5 \not{s}}{4m} \pm \frac{\not{p}}{4m} + \frac{\gamma_5 \not{s}}{4}$$

From the previous part we know that in the limit  $|\vec{p}| \rightarrow \infty$  we obtain  $s^\mu \rightarrow p^\mu/m$ . So we can calculate

$$\begin{aligned} \frac{1}{4} \pm \frac{\not{p} \gamma_5 \not{p}}{4m^2} \pm \frac{\not{p}}{4m} + \frac{\gamma_5 \not{p}}{4m} &= \frac{1}{4} \mp \frac{\gamma_5}{4} \pm \frac{\not{p}}{4m} + \frac{\gamma_5 \not{p}}{4m} \\ &= \frac{1}{4} (1 \mp \gamma_5) \pm (1 \pm \gamma_5) \frac{\not{p}}{4m}. \end{aligned}$$

- d) Zeigen Sie, dass für freie Elektron-Wellenfunktionen mit  $s^\mu$  und  $p^\mu$  gilt

$$u_\alpha(p, s) \bar{u}_\beta(p, s) = (\Lambda_+(p))_{\alpha\delta} (\Sigma(s))_{\delta\beta},$$

wobei  $\alpha$  und  $\beta$  hier Spinorindizes sind.

Solution (nicer solution)

We have the following relations for the spin projector and the solutions of the Dirac equation:

$$\begin{aligned}\Sigma(s)u(p, s) &= u(p, s), \\ \Sigma(s)u(p, -s) &= 0,\end{aligned}$$

where  $u(p, s)$  and  $u(p, -s)$  are the solutions with spin up and spin. We can rewrite this to

$$u^\dagger(p, s)\Sigma^\dagger(s) = u^\dagger(p, s),$$

With the relations  $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$  and  $\gamma_5^\dagger = \gamma_5$  we find  $\Sigma^\dagger(s) = \gamma^0\Sigma(s)\gamma^0$  and obtain the following relation

$$\begin{aligned}u^\dagger(p, s)\gamma^0\gamma^0\Sigma^\dagger(s)\gamma^0 &= u^\dagger(p, s)\gamma^0 \\ \rightarrow \bar{u}(p, s)\Sigma(s) &= \bar{u}(p, s).\end{aligned}$$

We consider

$$u_\alpha(p, s)\bar{u}_\beta(p, s) = (\Sigma(s)u(p, s))_\alpha (\bar{u}(p, s)\Sigma(s))_\beta.$$

Next we add a zero in the brackets by adding the spinors with  $-s$  which in combination of  $\Sigma(s)$  leads to zero:

$$u_\alpha(p, s)\bar{u}_\beta(p, s) = (\Sigma(s)(u(p, s) + u(p, -s)))_\alpha ((\bar{u}(p, s) + \bar{u}(p, -s))\Sigma(s))_\beta$$

Now we can use the completeness relations of the spinors

$$\Sigma_{r=\{s, -s\}} u(p, r)_\alpha \bar{u}(p, r)_\beta = \Lambda_+(p)_{\alpha\beta}.$$

Insert this in our expression leads to

$$\begin{aligned}u_\alpha(p, s)\bar{u}_\beta(p, s) &= (\Sigma(s)(u(p, s) + u(p, -s)))_\alpha ((\bar{u}(p, s) + \bar{u}(p, -s))\Sigma(s))_\beta \\ &= \Sigma_{\alpha\mu}(s) ((u(p, s) + u(p, -s)))_\mu ((\bar{u}(p, s) + \bar{u}(p, -s)))_\nu \Sigma_{\nu\beta}(s) \\ &= \Sigma_{\alpha\mu}(s)\Lambda_+(p)_{\mu\nu}\Sigma_{\nu\beta}(s) \\ &= (\Sigma(s)\Lambda_+\Sigma(s))_{\alpha\beta} \\ &= (\Lambda_+\Sigma(s)\Sigma(s))_{\alpha\beta} \\ &= (\Lambda_+\Sigma(s))_{\alpha\beta},\end{aligned}$$

where we used in the last two steps that the commutator of  $\Sigma(s)$  and  $\Lambda_+$  vanishes (see a and b) and the properties for projectors in general that  $P^2 = P$ .

Solution

We start with left hand side of the equation:

$$u_\alpha(p, s)\bar{u}_\beta(p, s) = \begin{pmatrix} \frac{p_0+m}{2m}\chi\chi^\dagger & -\chi\chi^\dagger\frac{p^i\sigma^i}{2m} \\ \frac{p^i\sigma^i}{2m}\chi\chi^\dagger & -\frac{|\vec{p}|^2}{2m}\chi\chi^\dagger \end{pmatrix}$$

For the right hand side we get

$$\left[ \frac{p^0\gamma^0 - p^i\gamma^i + m\mathbb{1}}{2m} \right] \left[ \frac{\mathbb{1} + \gamma_5 s_\mu \gamma^\mu}{2} \right] = \begin{pmatrix} \frac{p_0+m}{2m} & -\frac{p^i\sigma^i}{2m} \\ \frac{p^i\sigma^i}{2m} & -\frac{|\vec{p}|^2}{2m} \end{pmatrix} \begin{pmatrix} \frac{1}{2} + \frac{s^j\sigma^j}{2} & 0 \\ 0 & \frac{1}{2} - \frac{s^j\sigma^j}{2} \end{pmatrix}$$

Now we can consider for example a spin along the z-axis  $s^\mu = (0, 0, 0, 1)^T$  for which we obtain

$$\begin{pmatrix} \frac{1}{2} + \frac{s^j\sigma^j}{2} & 0 \\ 0 & \frac{1}{2} - \frac{s^j\sigma^j}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{\sigma^3}{2} & 0 \\ 0 & \frac{1}{2} - \frac{\sigma^3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

### Aufgabe 3: Gordon-Zerlegung

- a) Zeigen Sie, dass gilt:  $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$ , wobei  $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ .

Solution

We show

$$g^{\mu\nu} - i\sigma^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu.$$

- b) Gegeben seien die Impulsraum-Spinoren  $u(p_i)$  und  $\bar{u}(p_f)$ , die die Dirac-Gleichungen  $(\not{p}_i - m)u(p_i) = 0$  bzw.  $\bar{u}(p_f)(\not{p}_f - m) = 0$  erfüllen. Zeigen Sie, dass gilt

$$\bar{u}(p_f)\gamma^\mu u(p_i) = \bar{u}(p_f) \left[ \frac{(p_i + p_f)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p_f - p_i)_\nu}{2m} \right] u(p_i).$$

Solution

By using the relation of the first part we get

$$\begin{aligned} \bar{u}(p_f)\gamma^\mu u(p_i) &= \bar{u}(p_f) \left[ \frac{(p_i + p_f)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p_f - p_i)_\nu}{2m} \right] u(p_i) \\ &= \bar{u}(p_f) \left[ \frac{(p_i + p_f)^\mu}{2m} + \frac{g^{\mu\nu}(p_f - p_i)_\nu}{2m} - \frac{\gamma^\mu\gamma^\nu(p_f - p_i)_\nu}{2m} \right] u(p_i) \\ &= \bar{u}(p_f)\gamma^\mu u(p_i). \end{aligned}$$

#### Aufgabe 4: Foldy-Wouthuysen Transformation

Wir betrachten die Dirac-Gleichung für ein Elektron im Wasserstoffatom und verwenden den Ansatz

$$\psi = e^{-iEt} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}.$$

Daraus erhalten wir folgendes System von gekoppelten Differentialgleichungen

$$T \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \left[ -m + e\phi + \vec{\alpha} \cdot (\vec{p} - e\vec{A}) + m\beta \right] \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = H \begin{pmatrix} \varphi \\ \chi \end{pmatrix},$$

wobei  $T = E - m$  die kinetische Energie ist. Zeigen Sie, dass nach der Foldy-Wouthuysen Transformation mit der unitären Matrix  $U$

$$U = U^\dagger = C\beta + \frac{1}{2m}\vec{\alpha} \cdot \vec{p} \quad \text{mit} \quad C = \sqrt{1 - \frac{\vec{p}^2}{4m^2}},$$

die Diagonaleinträge für die obere Komponente  $\varphi$  aus dem Term

$$U\vec{\alpha} \cdot (\vec{p} - e\vec{A})U^\dagger$$

nach der Entwicklung bis einschließlich  $\mathcal{O}(\vec{p}^4/m^3)$  geschrieben werden können als

$$\frac{1}{2m} \left[ \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \vec{\sigma} \cdot \vec{p} + \vec{\sigma} \cdot \vec{p} \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \right] - \frac{\vec{p}^4}{8m^3},$$

wobei wir  $e|\phi|, e|\vec{A}| = \mathcal{O}(\vec{p}^2/m)$  annehmen.

Solution

We consider first the full expression

$$U\vec{\alpha} \cdot (\vec{p} - e\vec{A})U^\dagger = C^2\beta\vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \frac{C}{2m} \left[ \beta\vec{\alpha} \cdot (\vec{p} - e\vec{A}) \vec{\alpha} \cdot \vec{p} + \vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot (\vec{p} - e\vec{A}) \beta \right] + \frac{1}{4m^2}\vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot (\vec{p} - e\vec{A}) \vec{\alpha} \cdot \vec{p}.$$

The only terms that can contribute to the diagonal terms are even in  $\vec{\alpha}$ , so only the second will contribute to the diagonal terms. Using the relation  $\{\alpha^j, \beta\} = 0$ , we can write diagonal terms as

$$\begin{aligned} & \frac{C}{2m}\beta \left[ \beta\vec{\alpha} \cdot (\vec{p} - e\vec{A}) \vec{\alpha} \cdot \vec{p} + \vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot (\vec{p} - e\vec{A}) \beta \right] \\ &= \frac{C}{2m}\beta \begin{pmatrix} \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \vec{\sigma} \cdot \vec{p} + \vec{\sigma} \cdot \vec{p} \vec{\sigma} \cdot (\vec{p} - e\vec{A}) & 0 \\ 0 & \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \vec{\sigma} \cdot \vec{p} + \vec{\sigma} \cdot \vec{p} \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \end{pmatrix}. \end{aligned}$$

The diagonal term for the  $\varphi$  component is then given by

$$\begin{aligned} & \frac{\lambda}{2m} \left[ \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \vec{\sigma} \cdot \vec{p} + \vec{\sigma} \cdot \vec{p} \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \right] - \frac{\vec{p}^2}{8m^2} \frac{1}{m} (\vec{\sigma} \cdot \vec{p})^2 \\ &= \frac{\lambda}{2m} \left[ \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \vec{\sigma} \cdot \vec{p} + \vec{\sigma} \cdot \vec{p} \vec{\sigma} \cdot (\vec{p} - e\vec{A}) \right] - \frac{\vec{p}^4}{8m^3}, \end{aligned}$$

where we have expanded up  $\mathcal{O}(\vec{p}^4/m^3)$ .

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**Aufgabe 5: (\*) System von zwei identischen Teilchen (2 + 2 = 4 Punkte)**

Betrachten Sie ein System von zwei identischen Teilchen mit (i) Spin 1/2 und (ii) Spin 1, die jeweils ein von zwei Einteilchenzuständen annehmen können mit  $H|0\rangle = -\varepsilon|0\rangle$  und  $H|1\rangle = \varepsilon|1\rangle$ . Vernachlässigen Sie dabei die Wechselwirkung zwischen den beiden Teilchen.

- (a) Geben Sie die gemeinsamen Eigenvektoren der Operatoren  $\vec{S}^2$  und  $S_z$  an, wobei  $\vec{S}$  der Gesamtspinoperator ist.

*Hinweis:* Benutzen Sie dazu z.B. die tabellierten Clebsch-Gordan-Koeffizienten aus <https://pdg.lbl.gov/2018/reviews/rpp2018-rev-clebsch-gordan-coefs.pdf>.

Solution

- (i) The two spin-1/2 system ( $1/2 \oplus 1/2$ ) admits irreducible tensor representation ( $1 \otimes 0$ ), i.e. spin triplet and singlet. For the spin triplet we have symmetric states

$$|1, +1\rangle = \left| \frac{1}{2}, +\frac{1}{2} \right\rangle \left| \frac{1}{2}, +\frac{1}{2} \right\rangle = |\uparrow\rangle |\uparrow\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} \left( \left| \frac{1}{2}, +\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, +\frac{1}{2} \right\rangle \right) = \frac{1}{\sqrt{2}} (|\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle)$$

$$|1, -1\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = |\downarrow\rangle |\downarrow\rangle .$$

For the spin singlet we have one anti-symmetric state

$$|0, 0\rangle = \frac{1}{\sqrt{2}} \left( \left| \frac{1}{2}, +\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, +\frac{1}{2} \right\rangle \right) = \frac{1}{\sqrt{2}} (|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle) .$$

The coefficients in front of the  $|\uparrow\rangle |\uparrow\rangle, |\uparrow\rangle |\downarrow\rangle, |\downarrow\rangle |\uparrow\rangle, |\downarrow\rangle |\downarrow\rangle$  states can be read from Clebsch-Gordan table.

Solution

(ii) The two spin-1 system ( $1 \oplus 1$ ) admits irreducible tensor representation ( $2 \otimes 1 \otimes 0$ ). According to the Clebsch-Gordan table, we have the symmetric states for the spin quintuplet:

$$|2, +2\rangle = |1, +1\rangle |1, +1\rangle$$

$$|2, +1\rangle = \frac{1}{\sqrt{2}} (|1, +1\rangle |1, 0\rangle + |1, 0\rangle |1, +1\rangle)$$

$$|2, 0\rangle = \frac{1}{\sqrt{6}} |1, +1\rangle |1, -1\rangle + \sqrt{\frac{2}{3}} |0, 0\rangle |0, 0\rangle + \frac{1}{\sqrt{6}} |1, -1\rangle |1, +1\rangle$$

$$|2, -1\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle |1, -1\rangle + |1, -1\rangle |1, 0\rangle)$$

$$|2, -2\rangle = |1, -1\rangle |1, -1\rangle$$

and anti-symmetric states for spin triplet:

$$|1, +1\rangle = \frac{1}{\sqrt{2}} (|1, +1\rangle |1, 0\rangle - |1, 0\rangle |1, +1\rangle)$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} (|1, +1\rangle |1, -1\rangle - |1, -1\rangle |1, +1\rangle)$$

$$|1, -1\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle |1, -1\rangle - |1, -1\rangle |1, 0\rangle)$$

and symmetric state for spin singlet:

$$|0, 0\rangle = \frac{1}{\sqrt{3}} (|1, +1\rangle |1, -1\rangle - |1, 0\rangle |1, 0\rangle + |1, -1\rangle |1, +1\rangle) .$$

Note that the kets on the left hand side denotes the states in tensor representation.

- (b) Berechnen Sie die Energien und die Wellenfunktionen für alle möglichen Zustände. Geben Sie jeweils den Entartungsgrad an.

Solution

In total the system has three eigenenergies  $-2\varepsilon$ ,  $0$  and  $2\varepsilon$  with the eigenfunctions

$$|0\rangle_+ = |0\rangle |0\rangle,$$

$$|1\rangle_+ = \frac{1}{\sqrt{2}} (|1\rangle |0\rangle + |0\rangle |1\rangle)$$

$$|1\rangle_- = \frac{1}{\sqrt{2}} (|1\rangle |0\rangle - |0\rangle |1\rangle)$$

and

$$|2\rangle_+ = |1\rangle |1\rangle,$$

where subscript indicates whether the wavefunction is symmetric (+) or anti-symmetric (-).

(i) The two spin-1/2 system obeys fermionic statistics, hence its total wave functions are anti-symmetric. So taking into account the spin we get the following total wave functions

$$|0\rangle_+ |0, 0\rangle, \quad |1\rangle_+ |0, 0\rangle, |1\rangle_- |1, s\rangle, \quad |2\rangle_+ |0, 0\rangle.$$

Thus, the eigenenergies  $\pm 2\varepsilon$  are not degenerated and the eigenenergy  $0$  is 4 times degenerated.

Solution

(ii) The two spin-1 system obeys bosonic statistics, hence its total wave functions are symmetric. Thus, we have the following total wave functions

$$|0\rangle_+ |0, 0\rangle, |0\rangle_+ |2, 0\rangle, \quad |1\rangle_+ |0, 0\rangle, |1\rangle_+ |2, 0\rangle, |1\rangle_- |1, s\rangle \quad |2\rangle_+ |0, 0\rangle, |2\rangle_+ |2, s\rangle,$$

and the eigen energies  $\pm 2\varepsilon$  are 6 times degenerated and the eigen state  $0$  is 9 times degenerated.

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### Aufgabe 6: System von $N$ nicht-wechselwirkenden identischen Bosonen

Betrachten Sie ein System aus  $N$  nicht-wechselwirkenden identischen Bosonen, bei dem das  $i$ -te Niveau  $n_i$ -fach besetzt sei. Die Ein-Teilchen-Zustände seien gegeben durch  $|\alpha, i_\alpha\rangle$ , wobei  $\alpha$  bzw.  $i_\alpha$  den Teilchen- bzw. Niveauindex bezeichnen. Drücken Sie den Zustandsvektor dieses Systems unter Berücksichtigung des Symmetrisierungspostulats durch die Produktzustände

$$|1, i_1; 2, i_2; \dots; N, i_N\rangle = |1, i_1\rangle \otimes \dots \otimes |N, i_N\rangle$$

aus und leiten Sie die Normierungskonstante her.

Solution

Suppose we have the  $n_i$ -fold (degenerate) occupations that  $n_1 + n_2 + \dots + n_k = N$ , then we have  $\frac{N!}{n_1! n_2! \dots}$  possibilities to distribute the  $N$  bosons. Any distinct state can be described as  $|1, i_1\rangle \otimes |2, i_2\rangle \dots |N, i_N\rangle$ . Then the wave function is

$$\psi_B = \sqrt{\frac{n_1! n_2! \dots}{N!}} \sum |1, i_1\rangle \otimes |2, i_2\rangle \dots |N, i_N\rangle ,$$

where the summation sums over all possible distinct permutations of the level indices  $i_1, i_2, \dots, i_N$ .

### Aufgabe 7: Lagrange Multiplikatoren

Finden Sie das Maximum und Minimum der Funktion  $f(x, y) = x^2 + 2y^2 - x$  in

1.  $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ ;
2.  $K_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ .

Benutzen sie die Methode der Lagrange Multiplikatoren.

Solution

Since  $S^1$  and  $K_2$  are compact sets and  $f$  is continuous, it attains maximum and minimum both in  $S^1$  and in  $K_2$ .

1. The critical points of  $f(x, y) = x^2 + 2y^2 - x$  with the constraint  $g(x, y) = x^2 + y^2 - 1 = 0$  are determined through the equations  $\nabla f(x, y) = \lambda \nabla g(x, y)$  and  $g(x, y) = 0$ , i.e. by

$$\begin{cases} 2x - 1 = 2\lambda x \\ 4y = 2\lambda y \\ x^2 + y^2 - 1 = 0 \end{cases}$$

There are two cases: either  $y \neq 0$ , then  $\lambda = 2$  and we find the two critical points

$$P_1 = (-1/2, -\sqrt{3}/2) \quad \text{and} \quad P_2 = (-1/2, \sqrt{3}/2);$$

or  $y = 0$ , and then the critical points are

$$P_3 = (-1, 0) \quad \text{and} \quad P_4 = (1, 0)$$

(respectively for  $\lambda = 3/2$  and  $\lambda = 1/2$ ).

We observe that

$$f(-1/2, -\sqrt{3}/2) = f(-1/2, \sqrt{3}/2) = \frac{9}{4}, \quad f(-1, 0) = 2, \quad f(1, 0) = 0.$$

Hence  $f$  attains its maximum in the two critical points  $(-1/2, \pm\sqrt{3}/2)$ , its minimum in the critical point  $(1, 0)$ .

2. The set  $K$  can be decomposed as the union of  $S^1$  and the set  $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ . Critical points in the open set  $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$  are characterized by  $\nabla f(x, y) = (2x - 1, 4y) = 0$ . This equation is satisfied at  $(1/2, 0)$ . Since  $f(1/2, 0) = -1/4$ , we conclude that the maximum of  $f$  is attained on  $K$  in the two points  $(-1/2, \pm\sqrt{3}/2)$ , lying on the boundary of  $K$  (hence  $\max_{(x,y) \in K} f(x, y) = f(-1/2, \pm\sqrt{3}/2) = 9/4$ ). The minimum of  $f$ , on the other hand, is attained in the point  $(1/2, 0)$ , in the interior of  $K$  (hence,  $\min_{(x,y) \in K} f(x, y) = f(1/2, 0) = -1/4$ ).