Übungen zu Moderne Theoretische Physik III(TP)

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Exercise Sheet 1 SS-2024 Due date: 23.04.24

Born approximation (100 Points)

Exercise 1.1: *(40 points)* In the lectures, equation for elastic scattering amplitude

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\vec{r}\, e^{-i\vec{k}'\vec{r}} V(\vec{r})\psi(\vec{r}), \tag{1}$$

as well as a series solution to the function $\psi(\vec{r})$ in case when the potential $V(r)$ can be treated as a perturbation, were provided. These results imply that the amplitude can be written as

$$f(\vec{k}', \vec{k}) = \sum_{n=1}^{\infty} f^{(n)}, \tag{2}$$

where $f^{(n)}$ is proportional to n-th degree of potential, i.e. $f^{(n)} \sim V^n$.

(a) *(5 points)* The term $f^{(1)}$ is the amplitude in the Born approximation. Write it explicitly.

(b) *(10 points)* Show that $f^{(2)}$, which is the first correction to the amplitude in the Born approximation, can be written as

$$f^{(2)} = \frac{m^2}{\pi\hbar^4}\int \frac{\mathrm{d}^3\vec{l}}{(2\pi)^3}\frac{V_{\vec{k}',\vec{l}}\,V_{\vec{l},\vec{k}}}{\vec{l}^2 - \vec{k}^2 - i0}, \tag{3}$$

and

$$V_{\vec{k}_1,\vec{k}_2} = \int \mathrm{d}\vec{r}\, V(\vec{r})\, e^{-i\vec{r}\cdot(\vec{k}_1 - \vec{k}_2)}. \tag{4}$$

(c) *(10 points)* Optical theorem discussed in class states that the imaginary part of the forward scattering amplitude is proportional to the scattering cross section. In view of the reality of the Born amplitude $f^{(1)}$ this statement is obviously violated. Explain why.

(d) *(15 points)* Show that optical theorem is recovered if one considers the imaginary part of $f^{(2)}$ provided that the cross section is computed using the amplitude $f^{(1)}$. Make use of the following identity to compute the imaginary part

$$\mathrm{Im}\left[\frac{1}{x - i0}\right] = \pi\delta(x). \tag{5}$$

Exercise 1.2: *(45 points)* Compute in the Born approximation scattering amplitudes and total cross sections for the following potentials and discuss the range of parameters for which Born approximation for scattering on these potentials is applicable.

(a) *(15 points)*

$$U(r) = \alpha\delta(r - R), \tag{6}$$

(b) *(15 points)*

$$U(r) = U_0\, e^{-r/R}, \tag{7}$$

(c) *(15 points)*

$$U(r) = \begin{cases} U_0, & r < R \\ 0, & r > R. \end{cases} \tag{8}$$

Exercise 1.3: *(15 points)* For three potentials from **1.2** use the amplitudes which you have calculated and determine the cross sections in the limit of small $(kR \ll 1)$ and large $(kR \gg 1$ energies of the incoming particle.