Übungen zu Moderne Theoretische Physik III(TP)

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Exercise Sheet 3 | SS-2024 | Due date: 28.05.24

Theories with vector fields (100 Points)

Exercise 3.1: *(40 points)* Consider Lagrangian density of the free electromagnetic field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

(a) *(5 points)* We use system of units where $c = \hbar = 1$; the only dimensionfull quantity in this system is a mass. Find dimensions of x_μ, ∂_μ, $\mathcal{L}$, $F_{\mu\nu}$ and A_μ in units of mass.

(b) *(5 points)* Express components of the field-strength tensor $F_{\mu\nu}$ in terms of electric and magnetic fields using a relation between A^μ, $\vec{E}$ and $\vec{B}$ known from electrodynamics. Use obtained results to show that Lagrangian density in (1) can be written as

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right). \tag{2}$$

(c) *(10 points)* Derive a Hamiltonian operator from the Lagrangian density and write it in terms of electric and magnetic fields.

(d) *(10 points)* Modify a Lagrangian in (1)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e J^\mu A_\mu, \tag{3}$$

by adding an external current. Derive Euler-Lagrange equations. Show that the consistency of equations requires the current to be conserved,

$$\partial^\mu J_\mu = 0. \tag{4}$$

Demonstrate that field equations that follow from the above Lagrangian are equivalent to the pair of Maxwell's equations that contain electric charge density and current density.

(e) *(10 points)* Using explicit form of the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ show that Jacobi identity holds

$$\partial_\lambda F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0. \tag{5}$$

Use expressions for $F_{\mu\nu}$ in terms of electric and magnetic fields to show that the Jacobi identity is equivalent to two remaining Maxwell's equations.

Exercise 3.2: *(40 points)* Consider the theory of *massive vector field* described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu - e J^\mu A_\mu, \tag{6}$$

where J^μ is conserved current.

(a) *(15 points)* Derive field equations starting from the Lagrangian in Eq. (6). Show that consistency of equations *requires* $\partial_\mu A^\mu = 0$.

(b) *(25 points)* Study time-independent solutions of the field equations. Parametrise $J^\mu = (\rho_0(\vec{x}), \vec{j}(\vec{x}))$ and $A^\mu = (\phi, \vec{A})$ and express ϕ and $\vec{A}$ as integrals over $\rho(\vec{x})$ and $\vec{j}(\vec{x})$. It may be convenient to solve these equations by first considering a Fourier transform of both the current and charge densities as well as of ϕ and $\vec{A}$.

Exercise 3.3: *(20 points)* The most important aspect of the above solutions is the exponential suppression of vector potentials and fields at distances

$$r \sim 1/m, \tag{7}$$

away from the source. Hence, assuming that a particular interaction can be described by Lagrangian in Eq. (6), knowledge of interaction range allows us to estimate the mass of the relevant vector boson.

Perform such an analysis for the case of weak interactions where the effective range is known to be between 0.01 and $0.1\,\mathrm{fm}$ ($1\,\mathrm{fm} = 10^{-15}\mathrm{m}$). Provide an estimate of the (weak) vector boson mass in units of proton mass [1].

[1] See attached page, which also available for download from https://pdg.lbl.gov/2023/reviews/rpp2023-rev-phys-constants.pdf

1. Physical Constants

Table 1.1: Revised 2023 by D. Robinson (LBNL). Reviewed by P. Mohr (NIST). Mainly from "CODATA Recommended Values of the Fundamental Physical Constants: 2018," E. Tiesinga, D.B. Newell, P.J. Mohr, and B.N. Taylor, NIST SP961 (May 2019) [1]. The electron charge magnitude e, and the Planck, Boltzmann, and Avogadro constants h, k, and N_A, now join c as having defined values; the free-space permittivity and permeability constants ϵ_0 and μ_0 are no longer exact. These changes affect practically everything else in the Table. Figures in parentheses after the values are the 1-standard-deviation uncertainties in the last digits; the fractional uncertainties in parts per 10^9 (ppb) are in the last column. The full 2018 CODATA Committee on Data for Science and Technology set of constants are found at `https://physics.nist.gov/constants`. The last set of constants (beginning with the Fermi coupling constant) comes from the Particle Data Group. See also "The International System of Units (SI)," 9th ed. (2019) of the International Bureau of Weights and Measures (BIPM), `https://www.bipm.org/utils/common/pdf/si-brochure/SI-Brochure-9-EN.pdf`.

Quantity	Symbol, equation	Value	Uncertainty (ppb)
speed of light in vacuum	c	$299\,792\,458$ m s^{-1}	exact
Planck constant	h	$6.626\,070\,15\times10^{-34}$ J s (or J/Hz) §	exact
Planck constant, reduced	$\hbar \equiv h/2\pi$	$1.054\,571\,817\ldots \times 10^{-34}$ J s	exact*
		$= 6.582\,119\,569\ldots \times 10^{-22}$ MeV s	exact*
electron charge magnitude	e	$1.602\,176\,634\times10^{-19}$ C	exact
conversion constant	$\hbar c$	$197.326\,980\,4\ldots$ MeV fm	exact*
conversion constant	$(\hbar c)^2$	$0.389\,379\,372\,1\ldots$ GeV^2 mbarn	exact*
electron mass	m_e	$0.510\,998\,950\,00(15)$ MeV$/c^2 = 9.109\,383\,7015(28)\times10^{-31}$ kg	0.30
proton mass	m_p	$938.272\,088\,16(29)$ MeV$/c^2 = 1.672\,621\,923\,69(51)\times10^{-27}$ kg	0.31
		$= 1.007\,276\,466\,621(53)$ u $= 1836.152\,673\,43(11)$ m_e	0.053, 0.060
neutron mass	m_n	$939.565\,420\,52(54)$ MeV$/c^2 = 1.008\,664\,915\,95(49)$ u	0.57, 0.48
deuteron mass	m_d	$1875.612\,942\,57(57)$ MeV$/c^2$	0.30
unified atomic mass unit**	$u =$ (mass ^{12}C atom)/12	$931.494\,102\,42(28)$ MeV$/c^2 = 1.660\,539\,066\,60(50)\times10^{-27}$ kg	0.30
permittivity of free space	$\epsilon_0 = 1/\mu_0 c^2$	$8.854\,187\,8128(13)\ \times10^{-12}$ F m^{-1}	0.15
permeability of free space	$\mu_0/(4\pi\times10^{-7})$	$1.000\,000\,000\,55(15)$ N A^{-2}	0.15
fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	$7.297\,352\,5693(11)\times10^{-3} = 1/137.035\,999\,084(21)$† ‡‡	0.15
classical electron radius	$r_e = e^2/4\pi\epsilon_0 m_e c^2$	$2.817\,940\,3262(13)\times10^{-15}$ m	0.45
(e^- Compton wavelength)$/2\pi$	$\lambdabar_e = \hbar/m_e c = r_e\alpha^{-1}$	$3.861\,592\,6796(12)\times10^{-13}$ m	0.30
Bohr radius ($m_{\rm nucleus} = \infty$)	$a_\infty = 4\pi\epsilon_0\hbar^2/m_e e^2 = r_e\alpha^{-2}$	$0.529\,177\,210\,903(80)\times10^{-10}$ m	0.15
wavelength of 1 eV$/c$ particle	$hc/(1\ \text{eV})$	$1.239\,841\,984\ldots \times 10^{-6}$ m	exact*
Rydberg energy	$hcR_\infty = m_e e^4/2(4\pi\epsilon_0)^2\hbar^2 = m_e c^2\alpha^2/2$	$13.605\,693\,122\,994(26)$ eV	1.9×10^{-3}
Thomson cross section	$\sigma_T = 8\pi r_e^2/3$	$0.665\,245\,873\,21(60)$ barn	0.91
Bohr magneton	$\mu_B = e\hbar/2m_e$	$5.788\,381\,8060(17)\times10^{-11}$ MeV T^{-1}	0.30
nuclear magneton	$\mu_N = e\hbar/2m_p$	$3.152\,451\,258\,44(96)\times10^{-14}$ MeV T^{-1}	0.31
electron cyclotron freq./field	$\omega^e_{\rm cycl}/B = e/m_e$	$1.758\,820\,010\,76(53)\times10^{11}$ rad s^{-1} T^{-1}	0.30
proton cyclotron freq./field	$\omega^p_{\rm cycl}/B = e/m_p$	$9.578\,833\,1560(29)\times10^{7}$ rad s^{-1} T^{-1}	0.31
gravitational constant‡	G_N	$6.674\,30(15)\times10^{-11}$ m^3 kg^{-1} s^{-2}	2.2×10^4
		$= 6.708\,83(15)\times10^{-39}$ $\hbar c$ (GeV$/c^2)^{-2}$	2.2×10^4
standard gravitational accel.	g_N	$9.806\,65$ m s^{-2}	exact
Avogadro constant	N_A	$6.022\,140\,76\times10^{23}$ mol^{-1}	exact
Boltzmann constant	k	$1.380\,649\times10^{-23}$ J K^{-1}	exact
		$= 8.617\,333\,262\ldots \times 10^{-5}$ eV K^{-1}	exact*
molar volume, ideal gas at STP	$N_A k$ (273.15 K)/(101 325 Pa)	$22.413\,969\,54\ldots \times 10^{-3}$ m^3 mol^{-1}	exact*
Wien displacement law constant	$b = \lambda_{\rm max}T$	$2.897\,771\,955\ldots \times 10^{-3}$ m K	exact*
Stefan-Boltzmann constant	$\sigma = \pi^2 k^4/60\hbar^3 c^2$	$5.670\,374\,419\ldots \times 10^{-8}$ W m^{-2} K^{-4}	exact*
Fermi coupling constant‡‡	$G_F/(\hbar c)^3$	$1.166\,378\,8(6)\times10^{-5}$ GeV^{-2}	510
weak-mixing angle	$\sin^2\widehat{\theta}(M_Z)$ ($\overline{\rm MS}$)	$0.231\,21(4)$††	1.7×10^5
$W^\pm$ boson mass	m_W	$80.377(12)$ GeV$/c^2$¶	1.5×10^5
Z^0 boson mass	m_Z	$91.1876(21)$ GeV$/c^2$	2.3×10^4
strong coupling constant	$\alpha_s(m_Z)$	$0.1180(9)$	7.6×10^6

$\pi = 3.141\,592\,653\,589\,793\,238\ldots$		$e = 2.718\,281\,828\,459\,045\,235\ldots$	$\gamma = 0.577\,215\,664\,901\,532\,860\ldots$
$1\ \text{in} \equiv 0.0254$ m	$1\ \text{G} \equiv 10^{-4}$ T	$1\ \text{eV} = 1.602\,176\,634\times10^{-19}$ J (exact)	kT at 300 K $= [38.681\,727\,0718\ldots]^{-1}$eV (exact*)
$1\ \text{Å} \equiv 0.1$ nm	$1\ \text{dyne} \equiv 10^{-5}$ N	$(1\ \text{kg})c^2 = 5.609\,588\,603\ldots\times10^{35}$ eV(exact*)	$0\ ^\circ\text{C} \equiv 273.15$K
$1\ \text{barn} \equiv 10^{-28}$ m^2	$1\ \text{erg} \equiv 10^{-7}$ J	$1\ \text{C} = 2.997\,924\,58\times10^9$ esu	1 atmosphere $\equiv 760$ Torr $\equiv 101\,325$Pa

§CODATA recommends that the unit be J/Hz to stress that in $h = E/\nu$ the frequency ν is in cycles/sec (Hz), not radians/sec.
*These are calculated from exact values and are exact to the number of places given (*i.e.* no rounding).
**The molar mass of ^{12}C is 11.999 999 9958(36) g.
†At $Q^2 = 0$. At $Q^2 \approx m_W^2$ the value is $\sim 1/128$.
‡Absolute laboratory measurements of G_N have been made only on scales of about 1 cm to 1 m.
‡‡See the discussion in Ch. 10, "Electroweak model and constraints on new physics."
††The corresponding $\sin^2\theta$ for the effective angle is 0.23153(4).
¶See the "Mass and width of the W boson" review.

References

[1] E. Tiesinga *et al.*, Rev. Mod. Phys. **93**, 025010 (2021).