Übungen zu Moderne Theoretische Physik III(TP)

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Exercise Sheet 4 | SS-2024 | Due date: 11.06.24

Symmetries (100 Points)

Exercise 4.1: *(50 points)* An important symmetry that is required to describe physical phenomena is the Lorentz symmetry. For field theory, this symmetry implies that fields that we use to construct actions transform under Lorentz transformations in a well-defined way. Starting from this, one can prove that actions do not change when Lorentz transformations are performed.

Lorentz transformations of four-vectors are described by 4×4 matrices Λ satisfying $\Lambda^T g \Lambda = g$, where g is a Minkowski space metric tensor.

(a) *(10 points)* For a selected direction z there are two types of Lorentz transforms: rotations around the z-axis (R_z), preserving distances in the plane orthogonal to z $(x^2 + y^2)$ and boosts (B_z) in z direction which leave the interval $(t^2 - z^2)$ invariant. Using explicit form of matrices

$$R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & \sin\theta_z & 0 \\ 0 & -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_z = \begin{pmatrix} \cosh\beta_z & 0 & 0 & \sinh\beta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta_z & 0 & 0 & \cosh\beta_z \end{pmatrix}, \tag{1}$$

show that these matrices are valid Lorentz transforms.

(b) *(10 points)* Compute the product of two boosts and explain obtained result

$$B_z B_y - B_y B_z. \tag{2}$$

(c) *(15 points)* Lagrangians of theories we are studying in the course are Lorentz scalars, so it is important to study how different objects change under the Lorentz transform to combine them into invariant combinations properly.

It is important to distinguish transformation of coordinates and transformation of functions of coordinates. Consider a coordinate Lorentz transformation

$$x^{\prime,\mu} = \Lambda^\mu_\nu x^\nu. \tag{3}$$

Functions of x can be

- Lorentz scalars if they satisfy the following property

$$\varphi'(x') = \varphi(x) = \varphi(\Lambda^{-1} x'), \tag{4}$$

- Lorentz vectors in which case the following equation holds

$$V'^\mu(x') = \Lambda^\mu_\nu V^\nu(x) = \Lambda^\mu_\nu V^\nu(\Lambda^{-1} x'). \tag{5}$$

Use properties of Lorentz transformations and properties of fields to determine transformation properties of the following quantities

$$x_\mu\, y^\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad A^\mu \partial_\mu \varphi, \quad \partial_\mu A^\mu, \quad F^{\mu\nu}, \quad \partial_\mu F^{\mu\nu}, \quad F^{\mu\nu} F_{\mu\nu}, \quad T^{\mu\nu}, \tag{6}$$

where x, y are coordinate vectors, φ is a scalar field, A^μ is a vector gauge field, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $T^{\mu\nu}$ is the energy-momentum tensor of the scalar field derived in Eq. (8.43) in lectures.

(d) *(15 points)* Show that actions

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right],$$
$$S = \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - g_s J_\mu(x)A^\mu(x)\right], \tag{7}$$

are invariant under Lorentz transformations provided that $J^\mu(x)$ transforms as a vector. *Hint:* use $\det\Lambda = 1$ to calculate Jacobian.

Exercise 4.2: *(30 points)* Another important class of symmetries are internal symmetries of Lagrangian.

(a) *(15 points)* Consder model of several *interacting* scalar fields

$$\mathcal{L} = \sum_{i=1}^{3}\partial_\mu\varphi_i\partial^\mu\varphi_i - \sum_{i=1}^{3} m_i^2\varphi_i^2 - \phi_3\sum_{i=2}^{3}\lambda_i\varphi_i^2 - \left(\sum_{i=1}^{3} g_i\varphi_i^2\right)^2. \tag{8}$$

For certain values of m_i, λ_i and g_i, the Lagrangian will exhibit certain symmetries. Give examples of such cases and write down the symmetry transformations for each case.

(b) *(15 points)* Consider a model with Lagrangian

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger\,\partial^\mu\Phi - m^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2 \tag{9}$$

where Φ is a *doublet* of complex fields

$$\Phi = \begin{pmatrix}\eta_1 + i\eta_2\\ \eta_3 + i\eta_4\end{pmatrix}. \tag{10}$$

Write down equivalent Lagrangian in terms of real fields η_i. We can combine fields into vector $\vec{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)^T$ and consider transformations of fields $\vec{\eta}' = R\vec{\eta}$. Provide conditions on the matrix R for which Lagrangian stays invariant.

Exercise 4.3: *(20 points)* In the Abelian gauge theory with a complex scalar field with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger\,(D^\mu\varphi) - m^2\varphi^\dagger\varphi, \tag{11}$$

use Noether theorem to construct conserved current. Verify that in the limit of the zero gauge field obtained result is equal to one derived in lectures in the theory without gauge interaction.