Übungen zu Moderne Theoretische Physik III(TP)

V: Prof. Kirill Melnikov, U: Andrey Pikelner

Exercise Sheet 5 | SS-2024 | Due date: 25.06.24

Non-abelian gauge theories (100 Points)

Exercise 5.1: *(20 points)* When we talk about field theories, we refer to any term in a Lagrangian that involves more than two fields as an *interaction term* because such terms make field equations non-linear. Non-abelian gauge theories contain interaction terms even in the kinetic terms of gauge fields. Indeed, consider an $SU(2)$ gauge theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right], \tag{1}$$

where $\hat{F}_{\mu\nu} = F^a_{\mu\nu}\tau^a = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig\left[\hat{A}_\mu, \hat{A}_\nu\right]$, $\hat{A}_\mu = A^a_\mu \tau^a$ and $\left[\tau^a, \tau^b\right] = i\epsilon^{abc}\tau^c$.

(a) *(10 points)* Display all interaction terms that appear in Eq. (1).

(b) *(10 points)* In the system of units $\varepsilon_0 = \hbar = c = 1$ find dimension of the *coupling constant* g.

Exercise 5.2: *(20 points)* From the action constructed from the Lagrangian in Eq. (1) derive the field equations using the variational principle

$$\partial_\mu F^{a,\mu\nu} - g\epsilon^{abc}F^{b,\mu\nu}A^c_\mu = 0. \tag{2}$$

Exercise 5.3: *(10 points)* Use definition of of the covariant derivative $D^\mu = \partial^\mu - ig\hat{A}^\mu$, where $\hat{A}^\mu = A^{\mu,a}\tau^a$, to prove that

$$\hat{F}^{\mu\nu} \sim \left[D^\mu, D^\nu\right]. \tag{3}$$

Exercise 5.4: *(10 points)* Using definition of the covariant derivative and result obtained in the previous exercise, show that the correct form of the Jacobi identity in the non-abelian gauge theory has the following form

$$[D^\mu, \hat{F}^{\nu\lambda}] + [D^\lambda, \hat{F}^{\mu\nu}] + [D^\nu, \hat{F}^{\lambda\mu}] = 0. \tag{4}$$

Exercise 5.5: *(40 points)* Consider a theory of $SU(2)$ gauge fields and a fermion field ψ described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right] + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{5}$$

The field ψ transforms in a usual way under gauge transformations, $\psi \to U(x)\psi$, with $U(x) \in SU(2)$.

(a) *(20 points)* Construct conserved currents that exist in this theory using Noether theorem considering x-independent versions of gauge transformations. Derive equations of motion for the gauge fields in this theory and use it to prove that the derived currents are indeed conserved.

(b) *(20 points)* A simple generalization of the abelian fermion vector current to the non-abelian case is, obviously, $j^a_\mu = \bar{\psi}\tau^a\gamma_\mu\psi$. Show that this current satisfies the following equation

$$[D_\mu, J^\mu] = 0, \tag{6}$$

where

$$J_\mu = \sum_{a=1}^{3}\tau^a j^a_\mu. \tag{7}$$