Übungen zu Moderne Theoretische Physik III(TP)

V: Prof. Kirill Melnikov, U: Andrey Pikelner

Exercise Sheet 6 SS-2024 Due date: 09.07.24

Spontaneous symmetry breaking (100 Points)

Exercise 6.1: *(25 points)* Consider a theory of N complex scalar fields $\Phi = (\phi_1, \ldots, \phi_N)^T$ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^\dagger \partial^\mu \Phi + \frac{\mu^2}{2}\Phi^\dagger\Phi - \frac{\lambda}{4}\left(\Phi^\dagger\Phi\right)^2, \tag{1}$$

where $\Phi^\dagger\Phi = \sum_{i=1}^{N} \phi_i^* \phi_i$.

(a) *(5 points)* Show that this Lagrangian is invariant under $SU(N)$ transformations $\Phi' = U\Phi$, where U are $N \times N$ unitary matrices with $\det U = 1$.

(b) *(5 points)* Compute the Hamiltonian and write the formula for the total energy that can be stored in a field $\Phi(t, \vec{x})$. Assuming that $\mu^2 > 0$, determine the minimal value of energy that can be stored and the field configuration for which this happens.

(c) *(7 points)* Suppose that the energy is minimized for the constant field Φ_{vac} that satisfies the following condition

$$\Phi^+_{\mathrm{vac}}\Phi_{\mathrm{vac}} = v^2. \tag{2}$$

It is possible to satisfy the above equation by choosing Φ_{vac} in different ways. For example, we can choose the vacuum value of ϕ_N to be v and the value of all other components of the field Φ zero. Then, the arbitrary field is parametrized as

$$\Phi = \begin{pmatrix} \xi_1 \\ \vdots \\ v + \xi_N \end{pmatrix}, \tag{3}$$

and this time in the vacuum all complex $\xi_{1,2,..,N}$ fields vanish.

Write the Lagrangian of the theory after symmetry breaking in terms of ξ-fields, and explain which ξ fields get masses after the symmetry breaking and which do not. Determine the mass (or masses) of the field (or fields) that becomes massive.

(d) *(8 points)* Now repeat the above analysis by considering a different choice of the vacuum field, for example

$$\Phi = \begin{pmatrix} \xi_1 \\ \vdots \\ \frac{v}{\sqrt{2}} + \xi_{N-1} \\ \frac{v}{\sqrt{2}} + \xi_N \end{pmatrix}. \tag{4}$$

Write the Lagrangian in terms of the fields ξ, determine which fields become massive and which remain massless, and determine their masses. Compare the results of the calculations in this and in the previous item and explain your observations.

Exercise 6.2: *(30 points)* Consider a theory of three real-valued scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)^T$ and three $O(3)$ gauge fields $A_\mu^{1,2,3}$. The Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu,a} + D_\mu\phi_a D^\mu\phi_a + \frac{\mu^2}{2}\left(\vec{\phi}\cdot\vec{\phi}\right) - \frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2. \tag{5}$$

The covariant derivative is given by the following formula

$$D_\mu \phi_a = \partial_\mu \phi_a + g\epsilon_{abc} A_{\mu,b}\phi_c, \tag{6}$$

and the field-strength tensor $F^a_{\mu\nu}$ reads

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu. \tag{7}$$

(a) *(5 points)* Verify that the above Lagrangian possesses the $O(3)$ gauge symmetry corresponding to rotations of $\vec{\phi}$ and $\vec{A}_\mu$ vectors in field-space. To make the proof as simple as possible, it is useful to interpret various quantities in the Lagrangian that involve Levi-Civita tensor as vector products of field-space vectors.

(b) *(5 points)* Assuming that $\mu^2 > 0$, determine the energy of the ground-state field configuration in the above symmetry and write down equations that determine values of vacuum fields.

(c) *(10 points)* Consider two choices of the vacuum fields in the above theory

$$\vec{\phi}_{\rm vac,1} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \quad \vec{\phi}_{\rm vac,1} = \begin{pmatrix} \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{8}$$

Argue that an arbitrary field $\vec{\phi}$ can be written as

$$\vec{\phi}(x) = R(x)\vec{\phi}_{\rm vac}, \tag{9}$$

where $R(x)$ is the $O(3)$ rotation matrix. Pay particular attention to the number of independent parameters that are needed on the left-hand side and the right-hand side of the above equation.

Write the Lagrangian after the spontaneous symmetry breaking, explain what happens to matrices $R(x)$ in the above equation.

(d) *(10 points)* Use the above results to determine masses of gauge bosons in the two scenarios and masses of scalar fields that remain in the theory.

Exercise 6.3: *(45 points)* When discussing the Standard Model in lectures, it is assumed that electron neutrinos are massless. This was a reasonable assumption when the Standard Model was put together but it is known by now that (at least some) neutrinos must be massive. We will discuss how to include neutrino mass into the simplified version of the Standard Model discussed in the lecture.

(a) *(5 points)* In the version of the Standard Model discussed in class, we had left-handed electron, right-handed electron and left-handed neutrino. Which additional field is needed if we want to discuss the massive neutrino.

(b) *(20 points)* It is known that right-handed neutrino does not participate in weak interactions and has no electric charge. Use this information to determine the hypercharge of the right-handed neutrino.

(c) *(20 points)* We have seen that for electrons the mass term is generated by Yukawa Lagrangian

$$\mathcal{L} = -f_e \bar{\psi}_L \phi e_R + \text{h.c.}, \tag{10}$$

after electroweak symmetry breaking when the Higgs field ϕ receives conventional vacuum expectation value.

To use a similar mechanism to generate the mass term for neutrinos, we will require a similar term

$$\mathcal{L} = -f_\nu \bar{\psi}_L \phi_1 \nu_R + \text{h.c.} \tag{11}$$

Explain why ϕ_1 cannot be ϕ, for the purposes of generating the *mass term for neutrino*.

Consider choosing $\phi_1 = i\sigma_2\phi^*$. Determine gauge transformation rules for ϕ_1 and show that the Lagrangian

$$\mathcal{L} = -f_\nu \bar{\psi}_L \phi_1 \nu_R + \text{h.c.} \tag{12}$$

is invariant under $SU(2)$ and $U(1)$ gauge transformations of the Standard Model for this choice of ϕ_1. Compute the neutrino mass.