Moderne Physik für Lehramtskandidaten

Vorlesung: PD Dr. S. Gieseke – Übung: Dr. B. Agarwal

Lösung 2

Besprechung: Fr. 10.11.2023

Aufgabe 1: Identitäten der Vektoranalysis (8 P)

Zeigen Sie die Gültigkeit der folgenden Identitäten für die Funktionen f, g und die Vektorfelder $\boldsymbol{A}, \boldsymbol{B}$:

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f \tag{1}$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) \tag{2}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) = 0 \tag{3}$$
$$\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{A}\big) = 0 \tag{4}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{A}\big) - \boldsymbol{\nabla}^2\boldsymbol{A} \tag{5}$$

wobei wir den Laplace-Operator $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ benutzt haben.

Lösung 1:

So the students can prove these via components or if they know index notation. I have included both below

- **(1 P)** $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$

  Index Notation:

  $$\partial_i(fg) = f\partial_i g + g\partial_i f$$

  i.e. it directly follows from linearity of differential operator.

  Components:

  $$\begin{aligned}\boldsymbol{\nabla}(fg) &= \Big(\partial_x(fg), \partial_y(fg), \partial_z(fg)\Big) \\ &= \Big(f\partial_x g + g\partial_x f, f\partial_y g + g\partial_y f, f\partial_z g + g\partial_z f\Big) \\ &= \Big(f\partial_x g, f\partial_y g, f\partial_z g\Big) + \Big(g\partial_x f, g\partial_y f, g\partial_z f\Big) \\ &= f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f\end{aligned}$$

- **(2 P)** $\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$

Index Notation:

$$
\begin{aligned}
\partial_i\big(\epsilon_{ijk}A_jB_k\big) &= \epsilon_{ijk}\partial_i\big(A_jB_k\big) \\
&= \epsilon_{ijk}\big(A_j\partial_iB_k + B_k\partial_iA_j\big) \\
&= \epsilon_{ijk}A_j\partial_iB_k + \epsilon_{ijk}B_k\partial_iA_j \\
&= -\epsilon_{jik}A_j\partial_iB_k + (-1)^2\epsilon_{kij}B_k\partial_iA_j \\
&= -\epsilon_{ijk}A_i\partial_jB_k + \epsilon_{ijk}B_i\partial_jA_k \\
&= B_i\big(\epsilon_{ijk}\partial_jA_k\big) - A_i\big(\epsilon_{ijk}\partial_jB_k\big) \\
&= B_i\big[\boldsymbol{\nabla}\times\boldsymbol{A}\big]_i - A_i\big[\boldsymbol{\nabla}\times\boldsymbol{B}\big]_i
\end{aligned}
$$

where we have used the Levi-Civita tensor to represent the cross product, and the product rule in the second line. In the third line we used the anti-symmetry of the Levi-Civita tensor to cycle the indices, then in the fourth line we've relabelled the indices.

Components: Consider first the left-hand side:

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\big(\boldsymbol{A}\times\boldsymbol{B}\big) &= \boldsymbol{\nabla}\cdot\Bigg(A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x\Bigg) \\
&= \partial_x\big(A_yB_z - A_zB_y\big) + \partial_y\big(A_zB_x - A_xB_z\big) + \partial_z\big(A_xB_y - A_yB_x\big) \\
&= A_y\partial_xB_z + B_z\partial_xA_y - A_z\partial_xB_y - B_y\partial_xA_z + A_z\partial_yB_x + B_x\partial_yA_z \\
&- A_x\partial_yB_z - B_z\partial_yA_x + A_x\partial_zB_y + B_y\partial_zA_x - A_y\partial_zB_x - B_x\partial_zA_y \\
&= B_x\big(\partial_yA_z - \partial_zA_y\big) + B_y\big(\partial_zA_x - \partial_xA_z\big) + B_z\big(\partial_xA_y - \partial_yA_x\big) \\
&- A_x\big(\partial_yB_z - \partial_zB_y\big) - A_y\big(\partial_zB_x - \partial_xB_z\big) - A_z\big(\partial_xB_y - \partial_yB_x\big) \\
&= \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})
\end{aligned}
$$

where we used linearity in the third line, and then collected relevant terms in the fourth line.

- **(1 P)** $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) = 0$

  Index Notation:

$$
\begin{aligned}
\epsilon_{ijk}\partial_j\partial_kf &= \epsilon_{ijk}\partial_k\partial_jf \\
&= \epsilon_{ikj}\partial_j\partial_kf \\
&= -\epsilon_{ijk}\partial_j\partial_k = 0
\end{aligned}
$$

  where we exchanged the order of derivates in the first line, relabeled the indices in the second, then used the antisymmetric properties of the Levi-Civita tensor to swap the j, k indices. Since the original object is equal to the negative of itself, it must be zero.

  This is a general property of an antisymmetric tensor (e.g. the Levi-Civita tensor) contracted with objects that are symmetric in the indices.

  Components:

$$
\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) &= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ \partial_xf & \partial_yf & \partial_zf \end{vmatrix} \\
&= \big(\partial_y\partial_zf - \partial_z\partial_yf\big)\boldsymbol{e}_x + \big(\partial_z\partial_xf - \partial_x\partial_zf\big)\boldsymbol{e}_y + \big(\partial_x\partial_yf - \partial_y\partial_xf\big)\boldsymbol{e}_z = 0
\end{aligned}
$$

  where we've used the fact that $\partial_i\partial_j = \partial_j\partial_i$.

- **(2 P)** $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$

  Index Notation: Follows in exactly the same fashion as the previous argument:

$$\partial_i \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_i \partial_j A_k = 0$$

  since the Levi-Civita tensor is antisymmetric while the partial derivatives are symmetric in the i, j indices.

  Components:

$$\begin{aligned}\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) &= \Big(\partial_x, \partial_y, \partial_z\Big) \cdot \Big(\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x\Big) \\ &= \partial_x \partial_y A_z - \partial_x \partial_z A_y + \partial_y \partial_z A_x - \partial_y \partial_x A_z + \partial_z \partial_x A_y - \partial_z \partial_y A_x = 0\end{aligned}$$

- **(2 P)** $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$

  Index Notation:

$$\begin{aligned}\epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l A_m &= \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m \\ &= \Big(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\Big) \partial_j \partial_l A_m \\ &= \partial_m \partial_i A_m - \partial_j \partial_j A_i \\ &= \partial_i \partial_m A_m - \partial^2 A_i \\ &= \big[\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}\big]_i\end{aligned}$$

  where we have used the identity for two Levi-Civita tensors contracted across one index:

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

  Components:

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) &= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ \partial_y A_z - \partial_z A_y & \partial_z A_x - \partial_x A_z & \partial_x A_y - \partial_y A_x \end{vmatrix} \\ &= \Big(\partial_y \big[\partial_x A_y - \partial_y A_x\big] - \partial_z \big[\partial_z A_x - \partial_x A_z\big]\Big) \boldsymbol{e}_x + \Big(\partial_z \big[\partial_y A_z - \partial_z A_y\big] - \partial_x \big[\partial_x A_y - \partial_y A_x\big]\Big) \boldsymbol{e}_y \\ &\qquad + \Big(\partial_x \big[\partial_z A_x - \partial_x A_z\big] - \partial_y \big[\partial_y A_z - \partial_z A_y\big]\Big) \boldsymbol{e}_z \\ &= \Big(\partial_y \partial_x A_y + \partial_z \partial_x A_z - (\partial_y^2 + \partial_z^2) A_x\Big) \boldsymbol{e}_x + \Big(\partial_z \partial_y A_z + \partial_x \partial_y A_x - (\partial_z^2 + \partial_x^2) A_y\Big) \boldsymbol{e}_y \\ &\qquad + \Big(\partial_x \partial_z A_x + \partial_y \partial_z A_y - (\partial_x^2 + \partial_y^2) A_z\Big) \boldsymbol{e}_z\end{aligned}$$

  Notice that each component has a Laplacian like term, for example for the x component we have $\partial_y^2 + \partial_z^2$. Inserting the Laplacian, and adding back in the missing squared partial derivative we obtain:

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) &= \Big(\partial_y \partial_x A_y + \partial_z \partial_x A_z + \partial_x^2 A_x - \boldsymbol{\nabla}^2 A_x\Big) \boldsymbol{e}_x + \Big(\partial_z \partial_y A_z + \partial_x \partial_y A_x + \partial_y^2 A_y - \boldsymbol{\nabla}^2 A_y\Big) \boldsymbol{e}_y \\ &\qquad + \Big(\partial_x \partial_z A_x + \partial_y \partial_z A_y + \partial_z^2 A_z - \boldsymbol{\nabla}^2 A_z\Big) \boldsymbol{e}_z \\ &= \partial_x \big(\partial_x A_x + \partial_y A_y + \partial_z A_z\big) \boldsymbol{e}_x + \partial_y \big(\partial_x A_x + \partial_y A_y + \partial_z A_z\big) \boldsymbol{e}_y \\ &\qquad + \partial_z \big(\partial_x A_x + \partial_y A_y + \partial_z A_z\big) \boldsymbol{e}_z - \boldsymbol{\nabla}^2 \big(A_x \boldsymbol{e}_x + A_y \boldsymbol{e}_y + A_z \boldsymbol{e}_z\big) \\ &= \partial_x (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \boldsymbol{e}_x + \partial_y (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \boldsymbol{e}_y + \partial_z (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \boldsymbol{e}_z - \boldsymbol{\nabla}^2 \boldsymbol{A} \\ &= \Big(\partial_x, \partial_y \partial_z\Big) (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} \\ &= \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}\end{aligned}$$

Aufgabe 2: Kugel- und Zylinderkoordinaten (5 P)

Es ist oft einfacher, ein dreidimensionales physikalisches Problem in einem anderen Koordinatensystem zu formulieren, z.B. Kugelkoordinaten oder Zylinderkoordinaten. Die Koordinatentransformation vom kartesischen Koordinatensystem bis Zylinderkoordinaten als Vektorgleichung mit dem Ortsvektor $\boldsymbol{r}$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$$

gegeben ist und für Kugelkoordinaten ist das

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}.$$

Die entsprechende Volumenelemente sind dann:

$$\mathrm{d}^3r \coloneqq \mathrm{d}x\mathrm{d}y\mathrm{d}z = \mathrm{d}V = \begin{cases} \rho\cdot\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z & \text{für Zylinderkoordinaten} \\ r^2\sin\theta\cdot\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \quad = \quad r^2\mathrm{d}r\mathrm{d}(\cos\theta)\mathrm{d}\phi & \text{für Kugelkoordinaten} \end{cases}$$

Das einfachste Koordinatensystem hängt von der Symmetrie des physikalisches Problems ab, z.B. wenn wir einen Draht betrachten, dann ist es klar, dass wir Zylinderkoordinaten verwenden sollen. Wenn wir ein anderes Koordinatensystem verwenden, müssen wir darauf achten, dass der Nabla-Operator die Form

$$\boldsymbol{\nabla} = \boldsymbol{e}_\rho\frac{\partial}{\partial\rho} + \boldsymbol{e}_\theta\frac{1}{\rho}\frac{\partial}{\partial\theta} + \boldsymbol{e}_z\frac{\partial}{\partial z} \qquad \text{(Zylinderkoordinaten)}$$

$$\boldsymbol{\nabla} = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad \text{(Kugelkoordinaten)}$$

In dieser Form können Sie den Nabla-Operator als den Gradient-Operator verwenden. Leider ist das nicht das Ende der Geschichte. Die Divergenz-Operatoren sind:

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \qquad \text{(Zylinderkoordinaten)}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(A_\phi) \qquad \text{(Kugelkoordinaten)}$$

Die Rotation-Operatoren sind:

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \Big(\frac{1}{\rho}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\Big)\boldsymbol{e}_\rho + \Big(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\Big)\boldsymbol{e}_\theta + \frac{1}{\rho}\Big(\frac{\partial}{\partial\rho}(\rho A_\theta) - \frac{\partial A_\rho}{\partial\theta}\Big)\boldsymbol{e}_z \qquad \text{(Zylinderkoord.)}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{r\sin\theta}\Big(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\Big)\boldsymbol{e}_r + \frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\Big)\boldsymbol{e}_\theta + \frac{1}{r}\Big(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\Big)\boldsymbol{e}_\phi \qquad \text{(Kugelkoord.)}$$

(a) **(3 P)** Berechnen Sie (in Kugelkoordinaten)

(i) $\boldsymbol{\nabla}\cdot\boldsymbol{e}_r$, $\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{e}_r\big)$, $\boldsymbol{\nabla}\times\boldsymbol{e}_r$, $\boldsymbol{\nabla}\cdot\boldsymbol{e}_\phi$, $\boldsymbol{\nabla}\times\boldsymbol{e}_\theta$;

(ii) die Divergenz der Funktion

$$\boldsymbol{v} = r\cos\theta\boldsymbol{e}_r + r\sin\theta\boldsymbol{e}_\theta + r\sin\theta\cos\theta\boldsymbol{e}_\phi$$

(b) **(2 P)** Berechnen Sie die Divergenz und die Rotation der Funktion

$$\boldsymbol{v} = \rho(2+\sin^2\theta)\boldsymbol{e}_\rho + \rho\sin\theta\cos\theta\boldsymbol{e}_\theta + 3z\boldsymbol{e}_z$$

in Zylinderkoordinaten.

Lösung 2:

(a) **(3 P)** i.e. 0,5 P per solution

(i)

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{e}_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\cdot 1\big) = \frac{2}{r}\\
\boldsymbol{\nabla}\Big(\boldsymbol{\nabla}\cdot\boldsymbol{e}_r\Big) &= \boldsymbol{\nabla}\Big(\frac{2}{r}\Big) = \boldsymbol{e}_r\frac{\partial}{\partial r}\Big(\frac{2}{r}\Big) + 0 + 0 = -\frac{2}{r^2}\boldsymbol{e}_r\\
\boldsymbol{\nabla}\times\boldsymbol{e}_r &= \frac{1}{r\sin\theta}(0-0)\boldsymbol{e}_r + \frac{1}{r}\big(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(1)-0\big)\boldsymbol{e}_\theta + \frac{1}{r}\big(0-\frac{\partial}{\partial\theta}(1)\big)\boldsymbol{e}_\phi = 0\\
\boldsymbol{\nabla}\cdot\boldsymbol{e}_\phi &= 0+0+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(1) = 0\\
\boldsymbol{\nabla}\times\boldsymbol{e}_\theta &= \frac{1}{r\sin\theta}(0-0)\boldsymbol{e}_r + \frac{1}{r}(0-0)\boldsymbol{e}_\theta + \frac{1}{r}\big(\frac{\partial}{\partial r}(r\cdot 1)-0\big)\boldsymbol{e}_\phi = \frac{1}{r}\boldsymbol{e}_\phi
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 r\cos\theta\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta r\sin\theta\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\big(r\sin\theta\cos\theta\big)\\
&= \frac{1}{r^2}3r^2\cos\theta + \frac{1}{r\sin\theta}r2\sin\theta\cos\theta + 0\\
&= 3\cos\theta + 2\cos\theta = 5\cos\theta
\end{aligned}
$$

(b) **Divergence:**

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho\cdot\rho(2+\sin^2\theta)\big) + \frac{1}{\rho}\frac{\partial}{\partial\theta}\big(\rho\sin\theta\cos\theta\big) + \frac{\partial}{\partial z}(3z)\\
&= \frac{1}{\rho}2\rho(2+\sin^2\theta) + \frac{1}{\rho}\rho(\cos^2\theta-\sin^2\theta) + 3\\
&= 4 + 2\sin^2\theta + \cos^2\theta - \sin^2\theta + 3\\
&= 4 + \sin^2\theta + \cos^2\theta + 3 = 8.
\end{aligned}
$$

Rotation:

$$
\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{v} = &\Big(\frac{1}{\rho}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(\rho\sin\theta\cos\theta)\Big)\boldsymbol{e}_\rho + \Big(\frac{\partial}{\partial z}(\rho(2+\sin^2\theta)) - \frac{\partial}{\partial\rho}(3z)\Big)\boldsymbol{e}_\theta\\
&+ \frac{1}{\rho}\Big(\frac{\partial}{\partial\rho}(\rho^2\sin\theta\cos\theta) - \frac{\partial}{\partial\theta}(\rho(2+\sin^2\theta))\Big)\boldsymbol{e}_z\\
= &\frac{1}{\rho}\big(2\rho\sin\theta\cos\theta - \rho 2\sin\theta\cos\theta\big) = 0
\end{aligned}
$$

Aufgabe 3: Integralsätze von Stokes und Gauß (7 P)

(a) **(4 P)** *Der Gaußsche Satz* beschreibt den elektrischen Fluss durch ein geschlossene Fläche. Seien $\boldsymbol{a}(\boldsymbol{r})$ ein hinreichend oft differenzierbares Vektorfeld und V ein Volumen mit geschlossener Oberfläche ∂V, dann gilt:

$$\int_V (\boldsymbol{\nabla}\cdot\boldsymbol{a}(\boldsymbol{r}))\mathrm{d}^3 r = \oint_{\partial V}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{f}$$

Betrachten Sie eine Kugel mit Radius R, deren Mittelpunkt der Ursprung ist. Bestätigen Sie die Gültigkeit des Gaußschen Satzes für das Vektorfeld

$$\boldsymbol{v}_1 = r^2 \boldsymbol{e}_r = r\boldsymbol{r}.$$

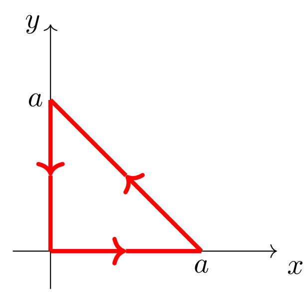
Gilt der Gaußsche Satz für das folgende Vektorfeld?

$$\boldsymbol{v}_2 = \frac{\boldsymbol{e}_r}{r^2}$$

(b) **(3 P)** *Der Stokes'sche Satz* - Seien $\boldsymbol{a}(\boldsymbol{r})$ ein hinreichend oft differenzierbares Vektorfeld und F eine Fläche mit dem Rand ∂F, dann gilt:

$$\int_F (\boldsymbol{\nabla} \times \boldsymbol{a}(\boldsymbol{r}))\mathrm{d}\boldsymbol{f} = \oint_{\partial F} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{r}$$

Betrachten Sie die rote dreieckige Fläche unten:



Testen Sie den Stokes'schen Satz für das Vektorfeld: $\boldsymbol{v} = (xy)\boldsymbol{e}_x + (2yz)\boldsymbol{e}_y + (3xz)\boldsymbol{e}_z$

Lösung 3:

(a) **(4 P)** Considering the vector field $\boldsymbol{v}_1 = r^2\boldsymbol{e}_r$, and using the spherical polar coordinates form of the divergence operator given in question 2, we begin by evaluating the left-hand side (LHS) of the divergence theorem:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{v}_1 &= \frac{1}{r}\frac{\partial}{\partial r}(r^2 \cdot r^2) + 0 + 0 = \frac{1}{r^2}4r^3 = 4r \\
\Rightarrow \int \mathrm{d}^3r\, \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 &= \int \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \cdot r^2 \sin\theta \cdot 4r \\
&= \underbrace{\left(\int_0^{2\pi} \mathrm{d}\phi\right)}_{2\pi} \underbrace{\left(\int_0^{\pi} \mathrm{d}\theta \sin\theta\right)}_{2} \left(\int_0^R \mathrm{d}r \cdot 4r^3\right) \\
&= 4\pi r^4\Big|_{r=R} = 4\pi R^4
\end{aligned}$$

For the right-hand side (RHS), we have to think about what defines the boundary of the volume we used to evaluate the LHS. We were considering a sphere of radius R, so the surface element should only be dependent on the solid angle of the volume element, and the normal to this surface is just the radial unit vector $\boldsymbol{e}_r = \hat{\boldsymbol{r}}$, i.e.:

$$\mathrm{d}\boldsymbol{f} = r^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi \cdot \boldsymbol{e}_r$$

Thus, the RHS becomes:

$$\int \mathrm{d}\boldsymbol{f} \cdot \boldsymbol{v}_1 = \int \Big(\mathrm{d}\theta \mathrm{d}\phi \, r^2 \sin\theta \boldsymbol{e}_r\Big) \cdot \big(r^2 \boldsymbol{e}_r\big)$$
$$= r^4 \underbrace{\Big(\int_0^{2\pi} \mathrm{d}\phi\Big)}_{2\pi} \underbrace{\Big(\int_0^{\pi} \mathrm{d}\theta \sin\theta\Big)}_{2}$$
$$= 4\pi r^4 \quad \Big(= 4\pi R^4 \text{ at } r = R\Big)$$

We now perform the same analysis for the vector field $\boldsymbol{v}_2 = \boldsymbol{e}_r/r^2$. LHS:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 \cdot \frac{1}{r^2}\big) + 0 + 0 = 0$$
$$\Rightarrow \int \mathrm{d}^3 r \boldsymbol{\nabla} \cdot \boldsymbol{v}_2 = 0$$

RHS:

$$\int \mathrm{d}\boldsymbol{f} \cdot \boldsymbol{v}_2 = \int \Big(\mathrm{d}\theta \mathrm{d}\phi \, r^2 \sin\theta \boldsymbol{e}_r\Big) \cdot \big(r^{-2} \boldsymbol{e}_r\big)$$
$$= \int \mathrm{d}\Omega = 4\pi$$

where we have employed a shorthand for the solid angle:

$$\mathrm{d}\Omega = \mathrm{d}\theta \mathrm{d}\phi \sin\theta$$

So in the case of $\boldsymbol{v}_2$, the divergence theorem breaks! The point is similar to Question 2b, part iii, namely that the divergence is zero everywhere *except at the origin*, where it diverges, meaning the calculation of the LHS is *incorrect*. The correct answer is 4π. The divergence theorem holds in general, except when you have divergences inside of the volume you are considering.

(b) **(3 P)** We begin with the LHS:

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ xy & 2yz & 3xz \end{vmatrix}$$
$$= \boldsymbol{e}_x\big(0 - 2y\big) + \boldsymbol{e}_y\big(0 - 3z\big) + \boldsymbol{e}_z\big(0 - x\big)$$
$$= \begin{pmatrix} -2y \\ -3z \\ -x \end{pmatrix}$$

The area element $\mathrm{d}\boldsymbol{f}$ only depends on x and y, and the normal direction is in the z direction:

$$\mathrm{d}\boldsymbol{f} = \mathrm{d}x\mathrm{d}y\boldsymbol{e}_z$$

Thus

$$\Big(\boldsymbol{\nabla} \times \boldsymbol{v}\Big) \cdot \mathrm{d}\boldsymbol{f} = -x\mathrm{d}x\mathrm{d}y$$

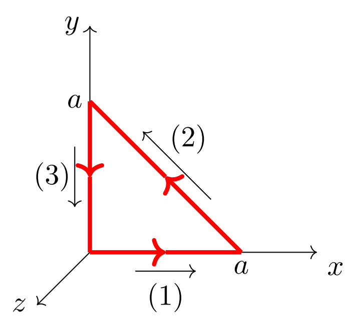
To integrate this, we have the following conditions: $0 \leq y, x \leq a$ and $y + x = a$. This allows us to write the integration limits in the second line below.

$$
\begin{aligned}
\int \mathrm{d}\boldsymbol{f} \cdot \left(\boldsymbol{\nabla} \times \boldsymbol{v}\right) &= -\int \mathrm{d}x \int \mathrm{d}y\, x \\
&= -\int_0^a \mathrm{d}y \int_0^{a-y} \mathrm{d}x\, x \\
&= -\frac{1}{2}\int_0^a \mathrm{d}y \left[x^2\right]_0^{a-y} \\
&= -\frac{1}{2}\int_0^a \mathrm{d}y \left(a^2 - 2ay + y^2\right) \\
&= -\frac{1}{2}\left[a^2 y - ay^2 + \frac{y^3}{3}\right]_0^a \\
&= -\frac{1}{2}\left[a^3 - a^3 + \frac{a^3}{3}\right] = -\frac{a^3}{6}
\end{aligned}
$$

To calculate the RHS, the line-element is simply:

$$
\begin{aligned}
\mathrm{d}\boldsymbol{l} &= \mathrm{d}x\boldsymbol{e}_x + \mathrm{d}y\boldsymbol{e}_y + \mathrm{d}z\boldsymbol{e}_z \\
\Rightarrow \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} &= (xy)\mathrm{d}x + (2yz)\mathrm{d}y + (3xz)\mathrm{d}z
\end{aligned}
$$

There are three segments:



and the integrand is then:

$$
\begin{aligned}
&(1) \rightarrow y = z = 0; \mathrm{d}y = \mathrm{d}z = 0; x : 0 \rightarrow a \\
&\quad \Rightarrow \int \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{v} = \int \mathrm{d}x(x \cdot 0) = 0
\end{aligned}
$$

$$
\begin{aligned}
&(2) \rightarrow z = 0; \mathrm{d}z = 0; y = a - x; x : a \rightarrow 0; \mathrm{d}y = -\mathrm{d}x \\
&\quad \Rightarrow \int \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{v} = \int_a^0 \mathrm{d}x\left(x(a-x) + 0 + 0\right) = \int_a^0 \mathrm{d}x(ax - x^2) \\
&\quad\quad = \left[\frac{ax^2}{2} - x^3/3\right]_a^0 = (0 - 0) - \left(\frac{a^3}{2} - \frac{a^3}{3}\right) = -\frac{a^3}{6}
\end{aligned}
$$

$$
\begin{aligned}
&(3) \rightarrow x = z = 0; \mathrm{d}x = \mathrm{d}z = 0; y : 0 \rightarrow a \\
&\quad \Rightarrow \int \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{v} = \int \mathrm{d}y(2y \cdot 0) = 0
\end{aligned}
$$

Adding them all up reproduces the LHS.