Moderne Physik für Lehramtskandidaten

Vorlesung: PD Dr. S. Gieseke – Übung: Dr. B. Agarwal

Lösung 12

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Aufgabe 1: Vektoren im Hilbertraum (7 P)

Die Vektoren $|v_1\rangle, |v_2\rangle$ bilden ein vollständiges Orthonormalsystem (VONS) in einem zweidimensionalen Hilbertraum $\mathcal{H}$, d.h. $\langle v_i|v_j\rangle = \delta_{ij}$. In Abhängigkeit dieser zwei Basisvektoren definieren wir die zwei Vektoren $|\phi\rangle, |\chi\rangle \in \mathcal{H}$ durch

$$
\begin{aligned}
|\phi\rangle &= (3-i)\,|v_1\rangle + (1+2i)\,|v_2\rangle \\
|\chi\rangle &= (1+i)\,|v_1\rangle + (1-i)\,|v_2\rangle
\end{aligned}
$$

(a) Berechnen Sie das Skalarprodukt $\langle\chi|\phi\rangle$. Zeigen Sie dann, dass die Vektoren

$$
\begin{aligned}
|u_1\rangle &= \frac{1}{\sqrt{2}}\,|v_1\rangle + \frac{i}{\sqrt{2}}\,|v_2\rangle \\
|u_2\rangle &= \frac{-i}{\sqrt{2}}\,|v_1\rangle - \frac{1}{\sqrt{2}}\,|v_2\rangle
\end{aligned}
$$

ebenfalls ein VONS bilden und bestimmen Sie die Komponenten von $|\phi\rangle$ und $|\chi\rangle$ bezüglich dieser neuen Basisvektoren.

Hinweis: For a vector $|A\rangle = \alpha\,|a\rangle$, we have $\langle A| = (\alpha\,|a\rangle)^\dagger = \alpha^*\,\langle a|$.

(b) Projektoren P_i auf Unterräume $\mathcal{H}_i$ haben die Eigenschaften $P_i^2 = P_i$ (Idempotenz) und $\sum_i P_i = 1$ (Vollständigkeit), falls die $\mathcal{H}_i$ den gesamten Raum $\mathcal{H}$ aufspannen. Betrachten Sie nun die Projektoren $P_{u_1} = |u_1\rangle\langle u_1|$ und $P_{v_1} = |v_1\rangle\langle v_1|$.

Welche mathematischen Objekte sind durch P_{u_1} bzw. P_{v_1} beschrieben? Bestimmen Sie die Komponenten $\langle v_j|\,P_{u_1}\,|v_k\rangle$ von P_{u_1} bezüglich der $|v_i\rangle$ und die Komponenten $\langle u_j|\,P_{v_1}\,|u_k\rangle$ von P_{v_1} bezüglich der $|u_i\rangle$. Schreiben Sie schließlich P_{u_1} in der Basis $|v_i\rangle$.

Lösung 1: Vectors in Hilbert space

(a) We consider the scalar product (c.f. dot product for ordinary 3D vectors):

$$
\begin{aligned}
\langle\chi|\phi\rangle &= \big(\langle v_1|\,(1-i) + \langle v_2|\,(1+i)\big)\big((3-i)\,|v_1\rangle + (1+2i)\,|v_2\rangle\big) \\
&= \langle v_1|v_1\rangle\,(1-i)(3-i) + \langle v_1|v_2\rangle\,(1-i)(1+2i) \\
&\quad + \langle v_2|v_1\rangle\,(1+i)(3-i) + \langle v_2|v_2\rangle\,(1+i)(1+2i)
\end{aligned}
$$

Using the orthogonality condition: $\langle v_i|v_j\rangle = \delta_{ij}$:

$$\begin{aligned}\langle\chi|\phi\rangle &= (1-i)(3-i) + (1+i)(1+2i)\\ &= 3 - i - 3i + i^2 + 1 + 2i + i + 2i^2\\ &= 1 - i\end{aligned}$$

To see if the proposed vectors $|u_i\rangle$ can form a complete set of basis vectors, we need them to obey the orthogonality condition: $\langle u_i|u_j\rangle = \delta_{ij}$. We consider the different combinations:

$$\begin{aligned}\langle u_1|u_1\rangle &= \Big(\frac{1}{\sqrt{2}}\Big)^2\big(\,\langle v_1| - i\,\langle v_2|\,\big)\big(\,|v_1\rangle + i\,|v_2\rangle\,\big)\\ &= \frac{1}{2}\big(1+1\big) = 1\\ \langle u_1|u_2\rangle &= \frac{1}{2}\big(\,\langle v_1| + i\,\langle v_2|\,\big)\big(-i\,|v_1\rangle - |v_2\rangle\,\big)\\ &= \frac{1}{2}(-i+i) = 0 = 0* = \langle u_2|u_1\rangle\\ \langle u_2|u_2\rangle &= \frac{1}{2}\big(i\,\langle v_1| - \langle v_2|\,\big)\big(-i\,|v_1\rangle - |v_2\rangle\,\big)\\ &= \frac{1}{2}(1+1) = 1\\ \Rightarrow \langle u_i|u_j\rangle &= \delta_{ij}\end{aligned}$$

In order to write the vectors $|\phi\rangle$ and $|\chi\rangle$ in the new basis vectors, we need to translate from $|v_i\rangle$ to $|u_i\rangle$.

$$\begin{aligned}|u_1\rangle + i\,|u_2\rangle &= \frac{1+1}{\sqrt{2}}\,|v_1\rangle \quad \Rightarrow\ |v_1\rangle = \quad \frac{1}{\sqrt{2}}(|u_1\rangle + i\,|u_2\rangle)\\ i\,|u_1\rangle + |u_2\rangle &= -\frac{1+1}{\sqrt{2}}\,|v_2\rangle \quad \Rightarrow\ |v_2\rangle = -\frac{1}{\sqrt{2}}(i\,|u_1\rangle + |u_2\rangle)\end{aligned}$$

Thus:

$$\begin{aligned}|\phi\rangle &= (3-i)\frac{1}{\sqrt{2}}\big(\,|u_1\rangle + i\,|u_2\rangle\,\big) - (1+2i)\frac{1}{\sqrt{2}}(i\,|u_1\rangle + |u_2\rangle)\\ &= \frac{1}{\sqrt{2}}\Big((5-2i)\,|u_1\rangle + i\,|u_2\rangle\Big)\\ |\chi\rangle &= (1+i)\frac{1}{\sqrt{2}}\big(\,|u_1\rangle + i\,|u_2\rangle\,\big) - (1-i)\frac{1}{\sqrt{2}}(i\,|u_1\rangle + |u_2\rangle)\\ &= \frac{-2+2i}{\sqrt{2}}\,|u_2\rangle\end{aligned}$$

(b) The projection operators P_{u_1} and P_{v_1} are given by

$$P_{u_1} = |u_1\rangle\,\langle u_1|\,, \quad P_{v_1} = |v_1\rangle\,\langle v_1|$$

Consider the following quantity:

$$\begin{aligned}\langle v_j| P_{u_1} |v_k\rangle &= \frac{1}{2} \langle v_j|u_1\rangle \langle u_1|v_k\rangle \\ &= \frac{1}{2}\big(\langle v_j|v_1\rangle + i \langle v_j|v_2\rangle \big)\big(\langle v_1|v_k\rangle - i \langle v_2|v_k\rangle \big) \\ &= \frac{1}{2}\big(\delta_{j1} + i\delta_{j2}\big)\big(\delta_{1k} - i\delta_{2k}\big) \\ &= \frac{1}{2}\big(\delta_{j1}\delta_{1k} - i\delta_{j1}\delta_{2k} + i\delta_{j2}\delta_{1k} - i^2\delta_{j2}\delta_{2k}\big) \\ &= \frac{1}{2}\big(\delta_{j1}(\delta_{1k} - i\delta_{2k}) + \delta_{j2}(\delta_{2k} + i\delta_{1k})\big)\end{aligned}$$

The mathematical object that is most useful to us is the 2D Levi-Civita tensor:

$$\epsilon_{ij} = \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \langle v_j| P_{u_1} |v_k\rangle = \frac{1}{2}\Big(d_{jk} - i\epsilon_{jk}\Big)$$

Similarly:

$$\begin{aligned}\langle u_j| P_{v_1} |u_k\rangle &= \frac{1}{2}\big(\langle u_j|u_1\rangle + i \langle u_j|u_2\rangle \big)\big(\langle u_1|u_k\rangle - i \langle u_1|u_k\rangle \big) \\ &= \frac{1}{2}\big(\delta_{jk} - i\epsilon_{jk}\big)\end{aligned}$$

Checking by components will yield exactly the above results

Aufgabe 2: Geladener Harmonischer Oszillator (13 P)

Betrachten Sie einen harmonischen Oszillator mit der Ladung e in einem konstanten elektrischen Feld E, der durch den Hamiltonoperator

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}\hat{X}^2 + eE\hat{X}$$

beschrieben wird.

(a) Zeigen Sie, dass $\hat{H}$ durch die Auf- und Absteigeoperatoren

$$\hat{b}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{P}$$
$$\text{und} \quad \hat{b} = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P}$$

in die Form

$$\hat{H} = \hbar\omega\big(\hat{b}^\dagger\hat{b} + \frac{1}{2} + \Delta\big)$$

gebracht werden kann. Der Störterm ist dabei gegeben durch

$$\Delta = \frac{eE}{\omega\sqrt{2\hbar\omega}}\big(\hat{b}^\dagger + \hat{b}\big)$$

(b) Bringen Sie den gestörten harmonischen Oszillator in *Diagonalform*. Nutzen Sie dabei einen Shift der Auf- und Absteigeoperatoren

$$\hat{a}^{(\dagger)} = \hat{b}^{(\dagger)} + c$$

mit $c \in \mathbb{R}$ und bestimmen Sie c so, dass der Hamiltonoperator in kanonischer Form vorliegt

$$\hat{H} = \hbar\omega\big(\hat{a}^\dagger \hat{a} + d\big)$$

Wie sieht das Energiespektrum des gestörten harmonischen Oszillators aus?

(c) Berechnen Sie die Ortsunschärfe ΔX des harmonischen Oszillators. *Tipp:* Für die Eigenzustände $|n\rangle$ von $\hat{H}$ gilt:

$$\begin{aligned} \hat{a}^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \text{und}\ \ a\,|n\rangle &= \sqrt{n}\,|n-1\rangle \end{aligned}$$

Lösung 2: Quantum harmonic oscillator
First, we begin with a recap of the simple harmonic oscillator, with Hamiltonian:

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}\hat{X}^2$$

Using the time-independent Schrödinger equation (TISE):

$$\big[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\big]\psi(x) = E\psi(x)$$

Dirac's Ansatz:

$$\begin{aligned} \hat{a} &:= \sqrt{\frac{m\omega}{2\hbar}}\big(\hat{X} + \frac{i\hat{P}}{m\omega}\big) \\ \hat{a}^\dagger &:= \sqrt{\frac{m\omega}{2\hbar}}\big(\hat{X} - \frac{i\hat{P}}{m\omega}\big) \end{aligned}$$

where the first is the annihilation (or lowering) operator, and the second the creation (or raising) operator. Rearranging gives us:

$$\begin{aligned} \hat{X} &= \sqrt{\frac{\hbar}{2m\omega}}\big(\hat{a}^\dagger + \hat{a}\big) \\ \hat{P} &= i\sqrt{\frac{m\omega\hbar}{2}}\big(\hat{a}^\dagger - \hat{a}\big) \end{aligned}$$

Can we rewrite the Hamiltonian only in terms of these operators? Yes!

$$\begin{aligned} \hat{H} &= \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}\hat{X}^2 \\ &= -\frac{1}{2m}\frac{m\omega\hbar}{2}(\hat{a}^\dagger - \hat{a})^2 + \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}(\hat{a}^\dagger + \hat{a})^2 \\ &= -\frac{\hbar\omega}{4}\big((\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2\big) + \frac{\hbar\omega}{4}\big((\hat{a}^\dagger)^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^2\big) \\ &= \frac{\hbar\omega}{2}\big(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\big) \\ &= \hbar\omega\big(\hat{a}^\dagger\hat{a} + \frac{1}{2}[\hat{a}, \hat{a}^\dagger]\big) \end{aligned}$$

where in the last line we have used $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}$. What is the commutator explicitly?

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \frac{m\omega}{2\hbar}\Big[\big(\hat{X} + \frac{i\hat{P}}{m\omega}\big), \big(\hat{X} - \frac{i\hat{P}}{m\omega}\big)\Big] \\
&= \frac{m\omega}{2\hbar}\Big([\hat{X}, \hat{X}] - \frac{i}{m\omega}[\hat{X}, \hat{P}] + \frac{i}{m\omega}[\hat{P}, \hat{X}] + \frac{1}{m^2\omega^2}[\hat{P}, \hat{P}]\Big) \\
&= \frac{m\omega}{2\hbar}\big(\frac{2i}{m\omega}[\hat{P}, \hat{X}]\big) = 1
\end{aligned}$$

where in the last equality we have used the fact that $[\hat{P}, \hat{X}] = -i\hbar$.

Thus:

$$\hat{H} = \hbar\omega\big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\big)$$

There are two components to the Hamiltonian. The first term is the occupation number operator: $\hat{N} = \hat{a}^\dagger\hat{a}$ and the second term is the ground state energy.

(Occupation) number operator:

$$\hat{N}\,|n\rangle = n\,|n\rangle$$

with the states $|n\rangle$ being eigenfunctions of the Hamiltonian with:

$$\hat{H}\,|n\rangle = \hbar\omega(n + 1/2)\,|n\rangle$$

We want to look for eigenfunctions of $\hat{N}$:

$$\begin{aligned}
[\hat{N}, \hat{a}] &= [\hat{a}^\dagger\hat{a}, \hat{a}] \\
&= [\hat{a}^\dagger, \hat{a}]\hat{a} + \hat{a}^\dagger[\hat{a}, \hat{a}] \\
&= -[\hat{a}, \hat{a}^\dagger]\hat{a} + 0 = -\hat{a} \\
\Rightarrow [\hat{N}, \hat{a}] &= -\hat{a}
\end{aligned}$$

and similarly:

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$$

Sidenote: $\hat{N}$ is a hermitian operator, i.e. it has real eigenvalues, and thus there exists an orthonormal basis for eigenstates:

$$\langle n|n'\rangle = \delta_{nn'}$$

and since we know $\hat{N}\,|n\rangle = n\,|n\rangle$, we may ask ourselves what occurs for $\hat{N}\hat{a}^\dagger\,|n\rangle$? Is $\hat{a}^\dagger\,|n\rangle$ an eigenstate of the number operator?

$$\begin{aligned}
\hat{N}\hat{a}^\dagger\,|n\rangle &= \big(\hat{a}^\dagger\hat{N} + [\hat{N}, \hat{a}^\dagger]\big)\,|n\rangle \\
&= \big(\hat{a}^\dagger n + \hat{a}^\dagger\big)\,|n\rangle \\
&= \hat{a}^\dagger(n+1)\,|n\rangle = (n+1)\hat{a}^\dagger\,|n\rangle
\end{aligned}$$

So yes, it is! It also implies the following:

$$|n+1\rangle = \alpha_{\text{norm}}\hat{a}^\dagger\,|n\rangle$$

What about $\hat{a}|n\rangle$?

$$\begin{aligned}\hat{N}\hat{a}|n\rangle &= \big(\hat{a}\hat{N} + [\hat{N},\hat{a}]\big)|n\rangle \\ &= \big(\hat{a}n - \hat{a}\big)|n\rangle \\ &= (n-1)\hat{a}|n\rangle\end{aligned}$$

It is too!

With these ladder operators and using the fact that they are eigenstates, we can define the ground state:

$$\hat{a}|0\rangle = 0$$

i.e. there is a point at which we can no longer use the lowering operator to lower the state.

We then define all the other states with the following normalizations:

$$\begin{aligned}\sqrt{n+1}\,|n+1\rangle &= \hat{a}^\dagger|n\rangle \\ \sqrt{n}\,|n-1\rangle &= \hat{a}|n\rangle \\ \Rightarrow |n\rangle &= \frac{1}{\sqrt{n!}}\big(\hat{a}^\dagger\big)^n|0\rangle\end{aligned}$$

To find the explicit form of the wavefunctions, in theory we need to solve for the groundstate (involves Hermite polynomials, see wikipedia for more information) and interate to get all the other states using the last equation above.

Take home message:

$$\begin{aligned}\hat{H} &= \hbar\omega\big(\hat{N} + \frac{1}{2}\big) \\ \hat{N}|n\rangle &= n|n\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle\end{aligned}$$

where we have a complete orthonormal basis $|n\rangle$ and quantized energies:

$$\begin{aligned}\hat{H}|n\rangle &= E_n|n\rangle \\ \Rightarrow E_n &= \hbar\omega(n + \frac{1}{2})\end{aligned}$$

—

For this question, we have that the ladder operators are given by

$$\begin{aligned}\hat{b}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ \text{und}\ \ \hat{b} &= \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P}\end{aligned}$$

So what are $\hat{X}$ and $\hat{P}$ in terms of the ladder operators?

$$\begin{aligned}\hat{X} &= \sqrt{\frac{\hbar}{2m\omega}}\big(\hat{b}^\dagger + \hat{b}\big) \\ \hat{P} &= i\sqrt{\frac{m\omega\hbar}{2}}\big(\hat{b}^\dagger - \hat{b}\big)\end{aligned}$$

Replacing these quantities in the Hamiltonian:

$$\begin{aligned}\hat{H} &= -\frac{1}{2m}\frac{m\omega\hbar}{2}(\hat{b}^\dagger - \hat{b})^2 + \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}(\hat{b}^\dagger + \hat{b})^2 + eE\sqrt{\frac{\hbar}{2m\omega}}(\hat{b}^\dagger + \hat{b}) \\ &= \hbar\omega\Big(\hat{b}^\dagger\hat{b} + \frac{1}{2} + \frac{eE}{\omega\sqrt{2\hbar m\omega}}(\hat{b}^\dagger + \hat{b})\Big)\end{aligned}$$

where we have used the same steps as before for the simple harmonic oscillator.

So let us define the number operator in the usual manner $\hat{N} = \hat{b}^\dagger\hat{b}$. If we define a shifted operator $\hat{a}^{(\dagger)} = \hat{b}^{(\dagger)} + c$, where $c \in \mathbb{R}$, can we rewrite the Hamiltonian in the form $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + d)$ where d is the ground-state energy?

Inserting the shifted operators into the Hamiltonian:

$$\begin{aligned}\hat{H} &= \hbar\omega\big((\hat{b}^\dagger + c)(\hat{b} + c) + d\big) \\ &= \hbar\omega\big(\hat{a}^\dagger\hat{a} + c(\hat{a}^\dagger + \hat{a}) + c^2 + d\big)\end{aligned}$$

Comparing the coefficients we obtain:

$$\begin{aligned}c &= \frac{eE}{\omega\sqrt{2\hbar m\omega}} \\ c^2 + d &= \frac{1}{2} \\ \Rightarrow \hat{H} &= \hbar\omega\big(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{e^2E^2}{2\hbar m\omega^3}\big)\end{aligned}$$

The number operator for the shifted ladder operators fulfill:

$$\hat{N}_b\,|n\rangle = \hat{b}^\dagger\hat{b}\,|n\rangle = n\,|n\rangle$$

i.e. $\hat{H}$ is diagonal in $|\rangle\, n$.

Energy eigenvalues:

$$\begin{aligned}E_n &= \langle\hat{H}\rangle = \langle n|\,\hat{H}\,|n\rangle \\ &= \langle n|\,\hbar\omega\big(n + \frac{1}{2} - \frac{e^2E^2}{2\hbar m\omega^3}\big) \\ &= \hbar\omega\big(n + \frac{1}{2} - \frac{e^2E^2}{2\hbar m\omega^3}\,\langle n|n\rangle \\ &= \hbar\omega\big(n + \frac{1}{2} - \frac{e^2E^2}{2\hbar m\omega^3}\end{aligned}$$

Position uncertainty: Inserting the shifted operators into the relationship between the ladder operators and $\hat{X}$:

$$\begin{aligned}\Delta\hat{X}^2 &= \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2 \\ &= \langle n|\,\hat{X}^2\,|n\rangle - \big(\,\langle n|\,\hat{X}\,|n\rangle\,\big)^2 \\ &= \frac{\hbar}{2m\omega}\,\langle n|\,\big(\hat{a}^\dagger + \hat{a} - 2c\big)^2\,|n\rangle - \frac{\hbar}{2m\omega}\,\langle n|\,\big(\hat{a}^\dagger + \hat{a} - 2c\big)\,|n\rangle^2\end{aligned}$$

First term is:

$$\langle\hat{X}^2\rangle = \frac{\hbar}{2m\omega}\,\langle n|\,\Big(\hat{a}^{\dagger,2} + \hat{a}^2 + 4c^2 - 4c\hat{a} + \hat{a}^\dagger\hat{a} - 4c\hat{a}^\dagger + \hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\Big)\,|n\rangle$$

Using the fact that:

$$\langle n|\,\hat{a}^{\dagger,m}\,|n\rangle = \frac{(n+m)!}{n!}\,\langle n|n+m\rangle = 0$$

and similarly with the lowering operator, the only parts of the first term that remain are:

$$\begin{aligned}\langle \hat{X}^2 \rangle &= \frac{\hbar}{2m\omega} \langle n| \left(4c^2 + 2\hat{a}^\dagger \hat{a} + [\hat{a}, \hat{a}^\dagger]\right) |n\rangle \\ &= \frac{1}{m\omega}(n + \frac{1}{2}) + \frac{e^2 E^2}{m^2 \omega^4}\end{aligned}$$

For the second term:

$$\begin{aligned}\langle \hat{X} \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle n| \left(\hat{a}^\dagger + \hat{a} - 2c\right) |n\rangle \\ &= -2\frac{eE}{\omega\sqrt{2\hbar m\omega}} \sqrt{\frac{\hbar}{2m\omega}} \langle n|n\rangle \\ &= -\frac{eE}{m\omega^2}\end{aligned}$$

where the raising and lowering operators will simply produce 0, and we have substituted in the value of c.

Thus the uncertainty is then:

$$\Delta \hat{X}^2 = \left(n + \frac{1}{2}\right)\frac{\hbar}{m\omega}$$

Note that the perturbation of the system caused by the electric field *does not* modify the overall uncertainty of the system (setting $E = 0$ would result in the same result)!