Aufgabe 1

a) $(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow \bar\psi(i\overleftarrow{\partial\!\!\!/} + m) = 0$

Begründung (nicht gefragt!) $\left\{ \begin{array}{l} \gamma^0(\gamma_\mu)^+\gamma^0 = \gamma_\mu \\ \psi^+(-i\overleftarrow{\partial}_\mu\gamma^{\mu+} - m) = 0 \,/\times\gamma^0 \end{array}\right. \Rightarrow \bar\psi(-i\overleftarrow{\partial\!\!\!/} - m) = 0$

b) $\gamma^\mu p\!\!\!/ \gamma_\mu = p_\alpha\gamma^\mu\gamma^\alpha\gamma_\mu = p_\alpha(2g^{\mu\alpha}\gamma_\mu - \gamma^\alpha\underbrace{\gamma^\mu\gamma_\mu}_{4}) = 2\overbrace{p^\mu\gamma_\mu}^{p\!\!\!/} - 4p\!\!\!/ = -2p\!\!\!/$

d) $1/2 \otimes 1/2$ $\quad S = S_1+S_2,\ S_1+S_2-1,\ \ldots\ S_1-S_2$

$S = 1 ; 0$

e) $1 \otimes 1/2$ $\quad S = 1+1/2 ; 1-1/2$; $S=3/2$ $\ S_z = 3/2 ; 1/2 ; -1/2 ; -3/2 ;$

$S = 3/2 ; 1/2$ $\qquad S=1/2$ $\ S_z = 1/2 ; -1/2 ;$

c) $Tr(p\!\!\!/\underbrace{\gamma^\mu p\!\!\!/\gamma_\mu}) \overset{(b)}{=} Tr[p\!\!\!/(-2p\!\!\!/)] = -2p^2 Tr(1\!\!1) = -8p^2$

oder $\overset{\shortparallel}{}$ $p_\alpha p_\beta\, Tr(\gamma^\alpha\gamma^\mu\underbrace{\gamma^\beta\gamma_\mu}) = p_\alpha p_\beta\, Tr[\gamma^\alpha(2g^{\mu\beta} - \gamma^\beta\underbrace{\gamma^\mu)\gamma_\mu}] =$

$= p_\alpha p_\beta\, Tr[2\gamma^\alpha\gamma^\beta - 4\gamma^\alpha\gamma^\beta] = -2p_\alpha p_\beta\, \underbrace{Tr(\gamma^\alpha\gamma^\beta)}_{g^{\alpha\beta}Tr 1\!\!1}$

$= -2p^2\cdot 4 = -8p^2$

f) $\psi(\vec r_1,\vec s_1;\vec r_2,\vec s_2;\vec r_3,\vec s_3) = \psi(\vec r_1)\psi(\vec r_2)\psi(\vec r_3)\,\chi_S(\vec s_1,\vec s_2,\vec s_3)$; χ_S = symm. Funkt. bzgl. $m_{s_1}, m_{s_2}, m_{s_3}$;

g) $|njm\rangle$ $\quad \vec J^2; J_z ;$ $\quad [A, J_z] = 0$ $\quad \langle njm'|A|njm\rangle \neq 0$

$\langle njm'|[A, J_z]|njm\rangle = 0$

$\overset{\shortparallel}{}(m-m')\underbrace{\langle njm'|A|njm\rangle}_{\neq 0} = 0.$ $\quad | \Rightarrow m = m'$

h) $\psi'(x') = S(\Lambda)\psi(x) \Rightarrow \psi^{+\prime} = \psi^+S^+$ $\quad ; \bar\psi = \psi^+\gamma^0 \Rightarrow$

$\Rightarrow \bar\psi' = \psi^+S^+\gamma^0 = \psi^+\gamma^0\underbrace{(\gamma^0S^+\gamma^0)}_{S^{-1}} = \bar\psi S^{-1}$

$\Rightarrow \bar\psi'\psi' = \bar\psi\underbrace{S^{-1}S}_{1\!\!1}\psi = \bar\psi\psi$ = L-inv.

(i') $\bar\gamma^5 = \gamma^0(\gamma^5)^+\gamma^0 = -\gamma^5$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$; $(\gamma^5)^+ = -i(\gamma^3)^+(\gamma^2)^+(\gamma^1)^+(\gamma^0)^+ = +i\gamma^3\gamma$

$= +i\underbrace{(-1)^3(-1)^2(-1)^1}_{+1}\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$

$\Rightarrow \bar\gamma^5 = \gamma^0\underbrace{\gamma^5\gamma^0} = \gamma^0(-\gamma^0\gamma^5) = -\gamma^5$

(j) $i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle$ (*) mit $|\psi,t\rangle = U(t,t_0)|\psi,t_0\rangle$

einsetzen in (*) $\Rightarrow$ $i\hbar\frac{\partial}{\partial t}U(t,t_0) = H(t)U(t,t_0)$

für $H = H(t)$ die formale Lsg.:

$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(t')U(t',t_0)\,dt' =$

$= T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}dt' H(t')\right]$; T = Zeitordnung

Aufgabe 2

Dirac-Gl. im elmg. Feld $\quad [\gamma^\mu(i\partial_\mu - \varepsilon A_\mu) - m]\psi = 0$ (**)

$A_\mu = \begin{pmatrix}\phi \\ -\vec A\end{pmatrix} \Rightarrow A_\mu \to A'_\mu = \begin{pmatrix}\phi - \frac{\partial\Lambda}{\partial t} \\ -\vec A - \vec\nabla\Lambda\end{pmatrix} = \begin{pmatrix}\phi \\ -\vec A\end{pmatrix} - \begin{pmatrix}\partial_t\Lambda \\ \vec\nabla\Lambda\end{pmatrix} = A_\mu$

$\Rightarrow [\gamma^\mu(i\partial_\mu - \varepsilon A'_\mu) - m]\underbrace{e^{i\varepsilon\Lambda}\psi}_{\psi'} \overset{?}{=} 0$

$= e^{i\varepsilon\Lambda}\{\gamma^\mu[i(\underbrace{i\varepsilon}_{-\varepsilon})\partial_\mu\Lambda + i\partial_\mu - \varepsilon A_\mu + \varepsilon\partial_\mu\Lambda] - m\}\psi =$

$= e^{i\varepsilon\Lambda}\underbrace{\{\gamma^\mu[i\partial_\mu - \varepsilon A_\mu] - m\}\psi}_{\overset{(**)}{=}0} = 0.$ ✓

Aufgabe 3 a) $E_{kin} = \sqrt{\vec p^{\,2}c^2 + m^2c^4} - mc^2 = mc^2\left(1 + \frac{\vec p^{\,2}}{2m^2c^2} - \frac{\vec p^{\,4}}{8m^4}\right.$

$= \frac{\vec p^{\,2}}{2m} - \frac{\vec p^{\,4}}{8m^3c^2} + \ldots$

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b) $H_{rel} = -\frac{\vec{p}^4}{8m^3c^2} = -\frac{1}{2mc^2}\left(\frac{\vec{p}^2}{2m}\right)^2 = -\frac{1}{2mc^2}\,T^2$; $H_0 = T + V(r)$

$= -\frac{1}{2mc^2}\left(H_0 - V(r)\right)^2 = -\frac{1}{2mc^2}\left(H_0^2 - H_0V - VH_0 + V^2\right) \Rightarrow$

$[H_{rel}, \vec{L}^2] = [H_{rel}, L_z] = [H_{rel}, \vec{S}^2] = [H_{rel}, S_z] = 0$

$\Rightarrow$ $\psi_{n\ell m, m_s} = R_{n\ell}\, Y_{\ell m}\, \xi_{m_s}$ sind Eigenzustände von H_{rel}.

Verschiebung der Grundzustandsenergie.

$\langle E^{(1)}_{rel}\rangle = \langle \psi_{n\ell m, m_s} | H_{rel} | \psi_{n\ell m, m_s}\rangle$ mit $n=1, \ell=0, \dots$

$$\begin{cases} H_0|\psi_{n\ell m, m_s}\rangle = E_n ; \quad E_n = -\frac{mc^2}{2}\frac{(Z\alpha)^2}{n^2} ; \quad \underline{Z=1.} \\ V(r) = -\frac{Z\alpha\hbar c}{r} \end{cases}$$

$$\Rightarrow \underline{\langle E^{(1)}_{rel}\rangle} = -\frac{1}{2mc^2}\left[E_n^2 - 2E_n(-Z\alpha\hbar c)\underbrace{\langle \tfrac{1}{r}\rangle}_{\frac{1}{a_0 n^2}} + (Z\alpha\hbar c)^2\underbrace{\langle \tfrac{1}{r^2}\rangle}_{\frac{1}{a_0^2 n^3(\ell+1/2)}}\right]$$

$$= -\frac{1}{2mc^2}\left[E_n^2 - 2E_n\left(-\frac{Z\alpha\hbar c}{n^2}\cdot\frac{mcZ\alpha}{\hbar}\right) + (Z\alpha\hbar c)^2\left(\frac{Z\alpha\cdot mc}{\hbar}\right)^2\frac{1}{n^3(\ell+1/2)}\right]$$

$$= -\frac{1}{2mc^2}\left[E_n^2 - 2E_n\cdot\left(-\frac{mc^2\,(Z\alpha)^2}{n^2}\right) - 2\underbrace{\left(-\frac{mc^2}{2}\frac{(Z\alpha)^2}{n^2}\right)}_{E_n}\left(\frac{mc^2\,(Z\alpha)^2}{n(\ell+1/2)}\right)\right]$$

$$= -\frac{1}{2mc^2}\,E_n\cdot(mc^2)\cdot\frac{(Z\alpha)^2}{n^2}\left[-\frac{1}{2} + 2 - \frac{2n}{\ell+1/2}\right] =$$

$$\boxed{= -E_n\,\frac{(Z\alpha)^2}{n^2}\left(\frac{3}{4} - \frac{n}{\ell+1/2}\right)} \;;$$

$$\Rightarrow \langle E_{rel}\rangle_{GZ} = -E_0\cdot(Z\alpha)^2\left(\frac{3}{4} - \frac{1}{1/2}\right) = +\frac{5}{4}(Z\alpha)^2E_0 = \frac{5}{4}\alpha^2E_0$$

$$= \frac{5}{4}\alpha^2\left(-\alpha^2\frac{mc^2}{2}\right) = \underline{-\frac{5}{8}\alpha^4\cdot(mc^2) < 0.}$$

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<u>Aufgabe 5</u>

a) $H_Z = -(\vec{\mu}_L + \vec{\mu}_S)\cdot\vec{B} = -\left(\frac{e}{2m}\vec{L} + g_e\frac{e}{2m}\vec{S}\right)\cdot\vec{B}$

$= -\frac{e}{2m}\underbrace{(\vec{L} + 2\vec{S})\cdot\vec{B}}_{(L_z+2S_z)B}$ $\quad g_e \approx 2$

$H = H_0 + H_Z$;

$H_0 = E_{kin} + V_{Coul}(r)$

$\Rightarrow [H, \vec{L}^2] = [H, L_z] = [H, \vec{S}^2] = [H, S_z] = 0 \Rightarrow H$ diagonal in der Basis $|n, \ell, \dots$

$E = E_n + \Delta E$; $\quad E_n = \frac{E_0}{n}$; $E_0 = -\frac{mc^2}{2}\cdot(Z\alpha)^2$;

$\Delta E = \frac{-eB}{2m}(m_\ell + 2m_s)\hbar$; $\begin{cases} m_s = \pm 1/2 \\ m_\ell = -\ell, -\ell+1, \dots, \ell \end{cases}$

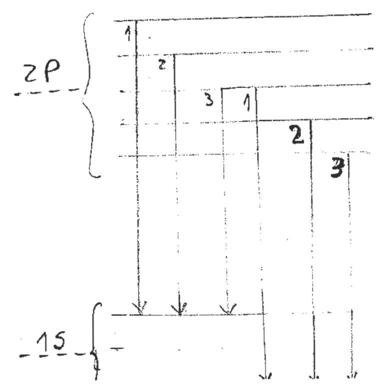
$= -\underbrace{\frac{eB}{2m}}_{>0}\underbrace{(m_\ell \pm 1)}_{M}\hbar$

- $n=1$, $\ell=0$, $m_\ell=0$; $m_s=\pm 1/2 \Rightarrow M = \begin{cases}+1\\-1\end{cases}$ nicht entartet
- $\underline{n=2}$, $\underline{\ell=0}$; $m_\ell=0$; $m_s=\pm 1/2 \Rightarrow M = \begin{cases}+1\\-1\end{cases}$
  - $\underline{\ell=1}$; $m_\ell=-1$; $m_s=\pm 1/2 \Rightarrow M = \begin{cases}0\\-2\end{cases}$
  - $m_\ell=0$; $m_s=\pm 1/2 \Rightarrow M = \begin{cases}+1\\-1\end{cases}$
  - $m_\ell=1$; $m_s=\pm 1/2 \Rightarrow M = \begin{cases}2\\0\end{cases}$

2-fach e... nic... e...

b) Auswahlregel f. elektrische Dipol. Übergänge:

$\begin{cases}\Delta\ell = \pm 1 \\ \Delta m_\ell = \pm 1, 0.\end{cases}$ Hier möglich $\begin{cases}\Delta\ell = +1; \ \Delta m_\ell = \pm 1; 0 \\ \Delta m_s = 0\end{cases}$



M	ℓ	m_ℓ	m_s
2	1	1	$+1/2$
1	1,0	0	$+1/2$
0	1	± 1	$\mp 1/2$
-1	1,0	0	$-1/2$
-2	1	-1	$-1/2$
1	0	0	$+1/2$
	0	0	$-1/2$

Es gibt 3 verschiedene Linien

(1) $\omega^{(1)} = \frac{E_2 - E_1}{\hbar} - \frac{eB}{2m}$ (3) $\omega^{(3)} = \frac{E_2 - E_1}{\hbar} + \frac{eB}{2m}$; $E_2 - E_1 = E_0$

(2) $\omega^{(2)} = \frac{E_2 - E_1}{\hbar}$

(c) $$R = \frac{\Gamma^{(1)}_{b\to a}}{\Gamma^{(3)}_{b\to a}} = \left(\frac{\omega^{(1)}}{\omega^{(3)}}\right)^3 \frac{|\langle b|\vec{r}|a\rangle^{(1)}|^2}{|\langle b|\vec{r}|a\rangle^{(3)}|^2}$$

$$|\langle b|\vec{r}|a\rangle^{(1)}| = |\langle \overset{n\,l\,m}{1,0,0}|\vec{r}|2,1,\pm 1\rangle|$$

$$|\langle b|\vec{r}|a\rangle^{(3)}| = |\langle 1,0,0|\vec{r}|2,1,\mp 1\rangle|$$

Spin variable nicht relevant für $\langle\vec{r}\rangle$

$\Rightarrow$ $|\langle b|\vec{r}|a\rangle^{(1)}| = |\langle b|\vec{r}|a\rangle^{(3)}|$ $|\langle b|\vec{r}|a\rangle|$ unabh. von m

- oder aus azimuthaler Symm.
- oder aus Wigner-Eckhart-Th.

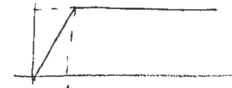
$\vec{r} \sim Y^{(1)}_m \Rightarrow \langle 1,0,0|Y^{(1)}_m|2,1,\pm 1\rangle =$

$$= \underbrace{\langle 1\pm 1, 1\mu|00\rangle}_{\text{Clebsch-Gordan}} \cdot \underbrace{\langle 21\| Y^{(1)} \| 10\rangle}_{\text{red. Matrix. Element}}$$

$\Rightarrow \mu = \mp 1$

$$\Rightarrow R = \left(\frac{\omega^{(1)}}{\omega^{(3)}}\right)^3 = \left(\frac{\frac{3}{4}|E_0| + \left|\frac{eB\hbar}{2m}\right|}{\frac{3}{4}|E_0| - \left|\frac{eB\hbar}{2m}\right|}\right)^3 = \left(\frac{3|E_0|/2 + |e|B\hbar/m}{3|E_0|/2 - |e|B\hbar/m}\right)^3.$$

Aufgabe 4 Sudden Approximation



t_s

$t_s \ll t_{ch}$ = char. Ze[...] Umordn[...]

- Ursprünglichen stationären Zust. von H_0:

$|m^{(0)}, t\rangle = e^{-iE_m^{(0)}t/\hbar}|m^{(0)}\rangle$

- Stationäre Zust von $H_0 + V$:

$|\bar{m}, t\rangle = e^{-i\bar{E}_m t/\hbar}|\bar{m}\rangle$

- $t \le 0$ $|\Psi, t\rangle = \phi_0(x) e^{-iE_0^{(0)}t/\hbar}$; $E_0^{(0)} = \frac{\hbar\omega}{2}$;

$t = 0_+$ $|\bar{\Psi}, 0\rangle = |\Psi, 0\rangle$ unmittelbar nach dem Einschalten [...] Potential befindet sich das Syst. in de[...]

$t > 0$: $V(x) = \frac{m\omega^2 x^2}{2} - qE\cdot x = \frac{m\omega^2}{2}\left(x^2 - \frac{2Eq}{m\omega^2}x + \frac{E^2q^2}{m^2\omega^4}\right) - \frac{E[...]}{m[...]}$

$= \frac{m\omega^2}{2}(x - x_0)^2 + V_0$ mit $x_0 = \frac{2(Eq)}{m\omega^2}$; $V_0 = -\frac{2[...]}{[...]}$

$H(x) = \frac{p^2}{2m} + \frac{m\omega^2}{2}(x-x_0)^2 + V_0 =$ $y = x - x_0$.

$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}(x-x_0)^2 + V_0$

$H(y) = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega^2}{2}y^2 + V_0$.

Gesucht $H(y)\bar{\Psi}(y) = \bar{E}\,\bar{\Psi}(y)$

$\Rightarrow \bar{\Psi}(y) = \phi_m(y)$; $\bar{E}_m = V_0 + E_m = V_0 + \hbar\omega([...]$

$$\Rightarrow \boxed{\bar{\Psi}_m(x) = \phi_m(x - x_0) = \mathscr{C}_n H_n\left(\frac{x-x_0}{\rho_0}\right) e^{-\frac{(x-x_0)^2}{2\rho_0^2}}}$$ $\mathscr{C}_n = ([...]$

$$\left\{\begin{array}{l} \bar{\Psi}(x,t) = \sum_m c_m(t)\,\bar{\Psi}_m(x) \quad \text{mit} \\ \bar{\Psi}(x, t=0) = \phi_0(x) \quad (t=0) \end{array}\right. \Big| \Rightarrow$$

$c_n(0) = \langle \phi_0(x)|\phi_n(x-x_0)\rangle = \int_{-\infty}^{+\infty} dx\, \phi_m(x-x_0)\phi_0^*(x) =$

$= \rho_0^{-1}(\pi 2^n n!)^{-1/2} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x}{\rho_0}\right)^2 - \frac{1}{2}\left(\frac{x-x_0}{\rho_0}\right)^2\right) H_n\left(\frac{x-x_0}{\rho_0}\right) d[...]$

$\xi = (x - x_0)/\rho_0 \Rightarrow$

$$c_n(0) \overset{(\text{Hermites})}{=} (\pi 2^n n!)^{-1/2} \int_{-\infty}^{+\infty} d\xi \, H_n(\xi) \exp\left[-\left(\xi^2 + \tfrac{1}{2}(x_0/\rho_0)^2 + \xi(-x_0/\rho_0)\right)\right]$$

$$= \frac{(-1)^n \left(\frac{x_0}{\rho_0}\right)^n \exp\left(-\frac{1}{4}\frac{x_0^2}{\rho_0^2}\right)}{\sqrt{2^n n!}} \quad ; \quad c_n(t) = c_n(0)\, e^{-i\bar{E}_n t/\hbar}$$

$$P_{\psi_0 \to \bar{n}} = |\langle \bar{n} | \psi, 0 \rangle|^2 = |c_n|^2 = |c_n(0)|^2 = \frac{\left(\frac{x_0}{\rho_0}\right)^{2n} \exp\left(\frac{-x_0^2}{2\rho_0^2}\right)}{2^n n!}$$

↳ Poisson Distribution